Fundamentals of compressed sensing for radio-interferometric imaging

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Introduction

Motivations

The probe of signals through Fourier measurements
is an acquisition strategy shared by various sensing techniques
in science and technology,
notably in astronomical and in biomedical imaging.

Presentation overview

(Introduction)

I. Radio interferometry

II. Compressed sensing

III. Spread spectrum technique

Conclusions

I.

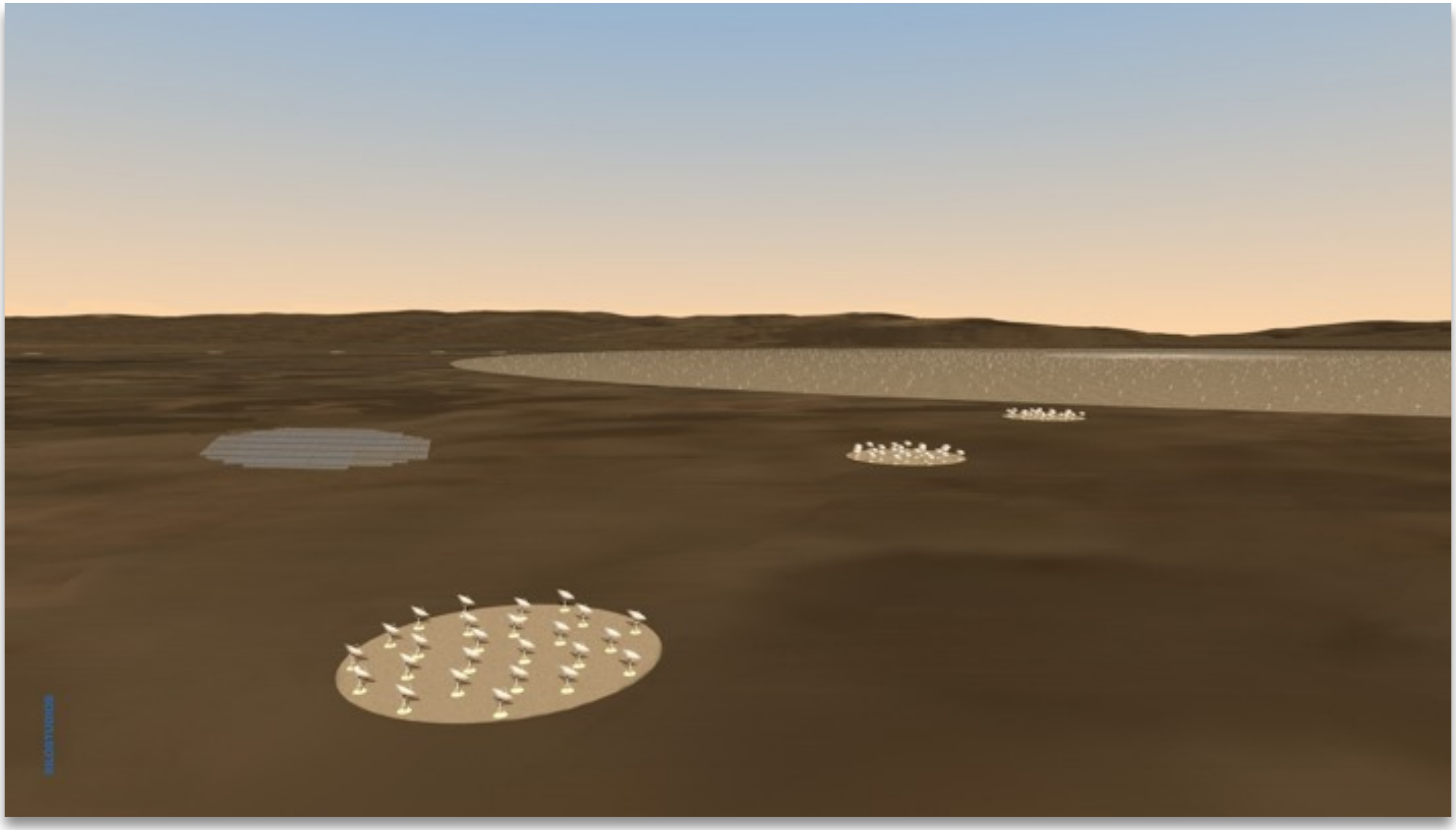
Radio interferometry

[1] A. R. Thompson, J. M. Moran, and G. W. Swenson Jr., "Interferometry and Synthesis in Radio Astronomy". WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2004.

Radio interferometry will play a major role in future astronomical observations, always seeking for better resolution and sensitivity on wider fields of view.

Telescopes

* Artist impression of the international SKA project... Australia or South africa?



SKA. Courtesy Xilostudios.

Basic inverse problem

* Under standard assumptions, visibilities provide noisy and incomplete Fourier coverage of a planar signal.

- Visibilities for small field of view:

$$y(\boldsymbol{k}_i) = \widehat{Ax}(\boldsymbol{k}_i).$$

- Ill-posed inverse (deconvolution) problem from $M/2 < N$ complex noisy visibilities for $\boldsymbol{x} \in \mathbb{R}^N$:

$$\boldsymbol{y} \equiv \boldsymbol{\Phi x} + \boldsymbol{n} \in \mathbb{C}^{(M/2)} \text{ with } \boldsymbol{\Phi} \equiv \mathsf{MFA} \in \mathbb{C}^{(M/2)\times N}.$$

- For Gaussian noise, the data constraints for candidate reconstruction $\bar{\boldsymbol{x}}$ reads in terms of a bound

$$\chi^2\left(\bar{\boldsymbol{x}}; \boldsymbol{\Phi}, \boldsymbol{y}\right) \equiv \sum_{r=1}^{M} \left[\frac{\left(\boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}}\right)_r}{\sigma^{(r)}}\right]^2 < \epsilon^2.$$

- Signal reconstruction algorithms differ through their regularization scheme. Standard iterative deconvolution algorithm: CLEAN, a Matching Pursuit algorithm.

II.
Compressed sensing

[1] E. J. Candès, J. Romberg, and T. Tao, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,” IEEE Trans. Inf. Theory, vol. 52, pp. 489-509, 2006.

[2] E. J. Candès, J. Romberg, and T. Tao, “Stable signal recovery from incomplete and inaccurate measurements,” Comm. Pure and Appl. Math., vol. 59, pp. 1207-1223, 2006.

[3] E. J. Candès, “Compressive sampling,” in Proc. Int. Congress Math, Madrid, 2006, vol. 3., pp. 1433-1452.

[4] D. L. Donoho, “Compressed sensing,” IEEE Trans. Inf. Theory, vol. 52, pp. 1289-1306, 2006.

[5] R. Baraniuk, “Compressive Sensing,” IEEE Signal Process. Mag., vol. 24, pp. 118-12, 2007.

[6] D. L. Donoho, J. Tanner, “Counting faces of randomly-projected polytopes when the projection radically lowers dimension,” J. Amer. Math. Soc., vol. 22, pp. 1-53, 2009.

Compressed sensing aims at merging data acquisition and compression for sparse or compressible signals.

It provides a framework to go beyond Nyquist-Shannon sampling theorem.

Sparsity and sensing

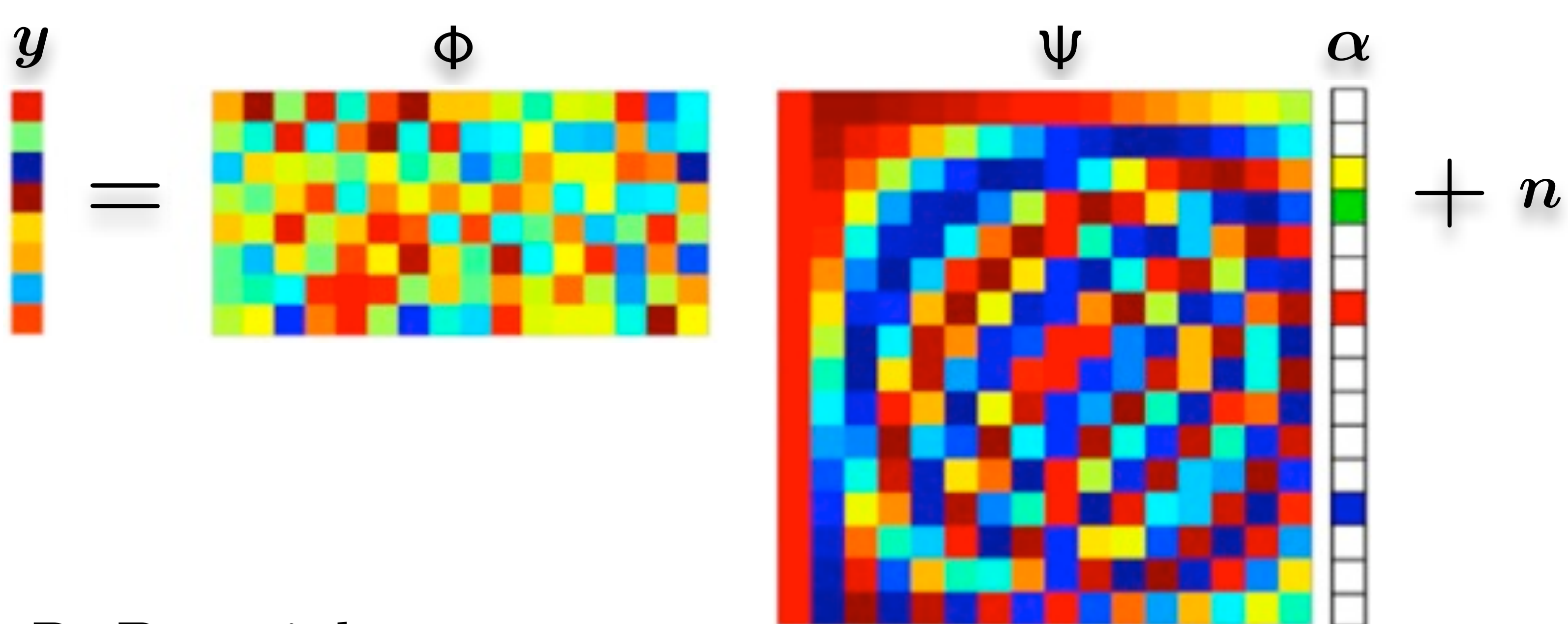
* Consider linear measurements of a sparse or compressible signal...

❖ Signal and sparsity basis: $\boldsymbol{x} \equiv \Psi \boldsymbol{\alpha} \in \mathbb{R}^N$ with $\Psi \in \mathbb{R}^{N \times N}$ o.n., for $\boldsymbol{\alpha} \in \mathbb{R}^N$ with $K \ll N$ non-zero or significant entries.

❖ Sensing matrix $\Phi \in \mathbb{R}^{M \times N}$ from $M < N$ real linear measurements.

❖ Independent and identically distributed Gaussian noise $\boldsymbol{n} \in \mathbb{R}^M$.

❖ Ill-posed inverse problem: $\boldsymbol{y} \equiv \Theta \boldsymbol{\alpha} + \boldsymbol{n} \in \mathbb{R}^M$ with $\Theta \equiv \Phi \Psi \in \mathbb{R}^{M \times N}$.



Images: courtesy R. Baraniuk

RIP and BP

* Randomness and incoherence are key for beating Nyquist-Shannon in reconstruction.

❖ Restricted Isometry Property (RIP) of order K : there exists some $\delta_K < 1$ such that, for all $\boldsymbol{\alpha}_K \in \mathbb{R}^N$ with K non-zero entries:

$$(1-\delta_K)\,||\boldsymbol{\alpha}_K||_2^2 \leq ||\Theta\boldsymbol{\alpha}_K||_2^2 \leq (1+\delta_K)\,||\boldsymbol{\alpha}_K||_2^2.$$

❖ With the RIP, accurate and stable reconstruction for the Basis Pursuit (BP) minimization problem seen as relaxation of "sparsity minimization":

$$\min_{\bar{\boldsymbol{\alpha}}\in\mathbb{R}^N} ||\bar{\boldsymbol{\alpha}}||_1 \text{ subject to } ||\boldsymbol{y}-\Theta\bar{\boldsymbol{\alpha}}||_2 \leq \epsilon. \qquad (\mathrm{BP}_\epsilon)$$

❖ Randomness in $\mathbf{\Phi}$ and incoherence with $\mathbf{\Psi}$ ensure the RIP at high probability, under a typical condition: $M \propto K \ll N$, hence the terminology "compressed sensing" !

❖ Random selection of Fourier measurements would do it, i.e. $\mathbf{\Phi} \equiv \mathsf{MF} \in \mathbb{R}^{M\times N}$:

$$K \leq \frac{cM}{N\mu^2(\mathsf{F},\mathbf{\Psi})\ln^4 N} \qquad \text{for mutual coherence} \quad \mu(\mathsf{F},\mathbf{\Psi}) \equiv \max_{1\leq i,j\leq N} |\boldsymbol{f}_i \cdot \boldsymbol{\psi}_j|.$$

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... Compressed sensing a posteriori justifies both the acquisition strategy (Fourier space is incoherent with real space) and the reconstruction procedure (CLEAN is equivalent to BP) used in radio interferometry.

But the theory goes much beyond, both at the level of acquisition and reconstruction...

[Versatile regularization]

[1] Y. Wiaux, L. Jacques, G. Puy, A. M. M. Scaife, and P. Vandergheynst, "Compressed sensing imaging techniques for radio interferometry," Mon. Not. R. Astron. Soc., vol. 395, pp. 1733-1742, 2009.

[2] Y. Wiaux, L. Jacques, G. Puy, A. M. M. Scaife, and P. Vandergheynst, "Compressed sensing for radio interferometry: prior-enhanced Basis Pursuit imaging techniques," in Proc. INRIA SPARS'09 Conf., Saint-Malo, 2009, pp. 00369428.

[3] Y. Wiaux, G. Puy, and P. Vandergheynst, "Compressed sensing reconstruction of a string signal from interferometric observations of the cosmic microwave background," Mon. Not. R. Astron. Soc., vol. 402, pp. 2626-2636, 2010.

IV.
Spread spectrum technique

[1] Y. Wiaux, L. Jacques, G. Puy, A. M. M. Scaife, and P. Vandergheynst, "Compressed sensing imaging techniques for radio interferometry," Mon. Not. R. Astron. Soc., vol. 395, pp. 1733-1742, 2009.

[2] Y. Wiaux, G. Puy, Y. Boursier, and P. Vandergheynst, "Spread spectrum for imaging techniques in radio interferometry," Mon. Not. R. Astron. Soc., 2009, vol. 400, pp. 1029-1038.

[3] G. Puy, Y. Wiaux, R. Gruetter, J.-Ph. Thiran, D. Van de Ville, and P. Vandergheynst, "Spread spectrum for interferometric and magnetic resonance imaging," in Proc. IEEE Int. Conf. on Acoustic, Speech and Signal Process. (ICASSP), Vol. CFP10ICA-CDR (2010). IEEE Signal Process. Soc., 2802.

What about the dependence of the signal reconstruction quality as a function of the sparsity basis (through the coherence)?

Setting up the problem...

* Fornax A radio emission around the elliptical galaxy NGC 1316.

Original image: $N = 512 \times 512$

Courtesy NRAO & Uson

Setting up the problem...

* Reconstruction of the Fornax A radio emission around the elliptical galaxy NGC 1316.

Image: $N = 512 \times 512$

Simulated acquisition: 30 dB noise, **random coverage of 10%**

Reconstruction: TV_ϵ minimization problem (assumes sparsity of the gradient)

$\mathrm{SNR} = 0.278\,\mathrm{dB}$

Spread spectrum

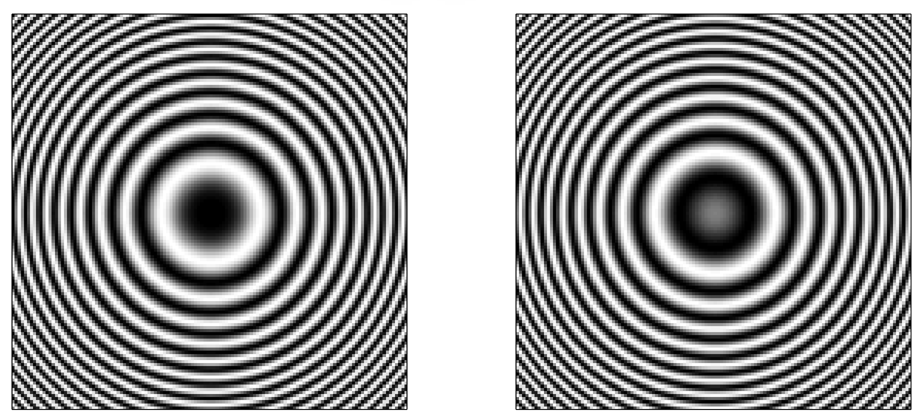
* The seeds for a modulation spreading the spectrum of the signal are present.

❖ In specific, and non-standard array configurations, a linear chirp modulation imprints the signal:

$$y(\boldsymbol{k}_i) \equiv \int_{\mathrm{D}_\tau} C^{(w)}(|\boldsymbol{\tau}|)\, Ax(\boldsymbol{\tau})\, \mathrm{e}^{-2\mathrm{i}\pi \boldsymbol{k}_i \cdot \boldsymbol{\tau}}\, \mathrm{d}\boldsymbol{\tau} \equiv \widehat{C^{(w)} Ax}(\boldsymbol{k}_i),$$

$$C^{(w)}(|\boldsymbol{l}|) = \mathrm{e}^{\mathrm{i}\pi w |\boldsymbol{l}|^2}$$

Real part



Imag. part

❖ The modulation amounts to a (norm-preserving) convolution in Fourier space, hence spreading the spectrum of the signal, hence decreasing the coherence:

$$\widehat{C^{(w)} Ax} = \widehat{C^{(w)}} \star \widehat{Ax}.$$

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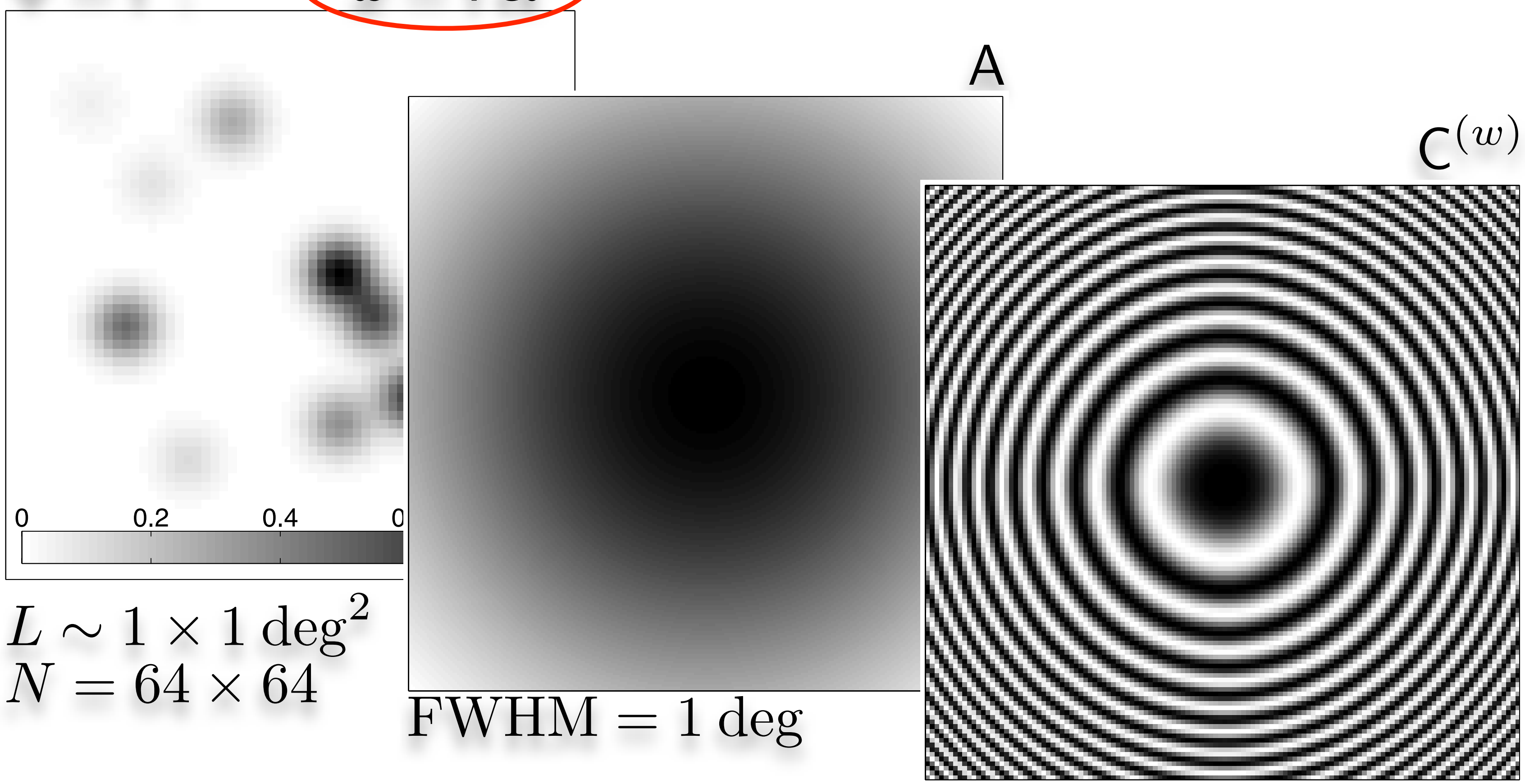
Simulations

* Chain of visibility simulations for signals made up of 10 Gaussian waveforms:

$\Psi = \Gamma$: $\boldsymbol{x} = \Gamma \boldsymbol{\alpha}$

A

$C^{(w)}$



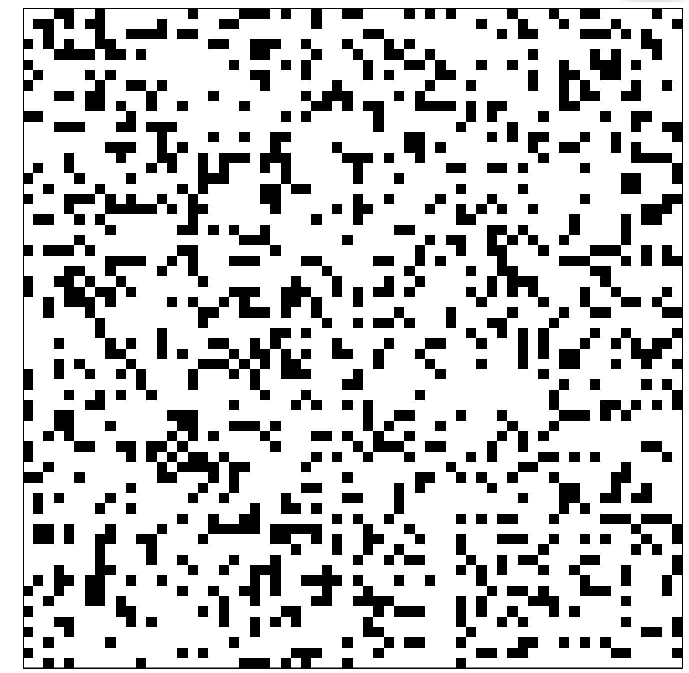
$L \sim 1 \times 1\ \mathrm{deg}^2$
$N = 64 \times 64$

FWHM $= 1$ deg

$wL^{1/2}/2 \simeq B$ (constant)

F

M



$M/2 \in [50, 1000]$

+ noise at 30dB

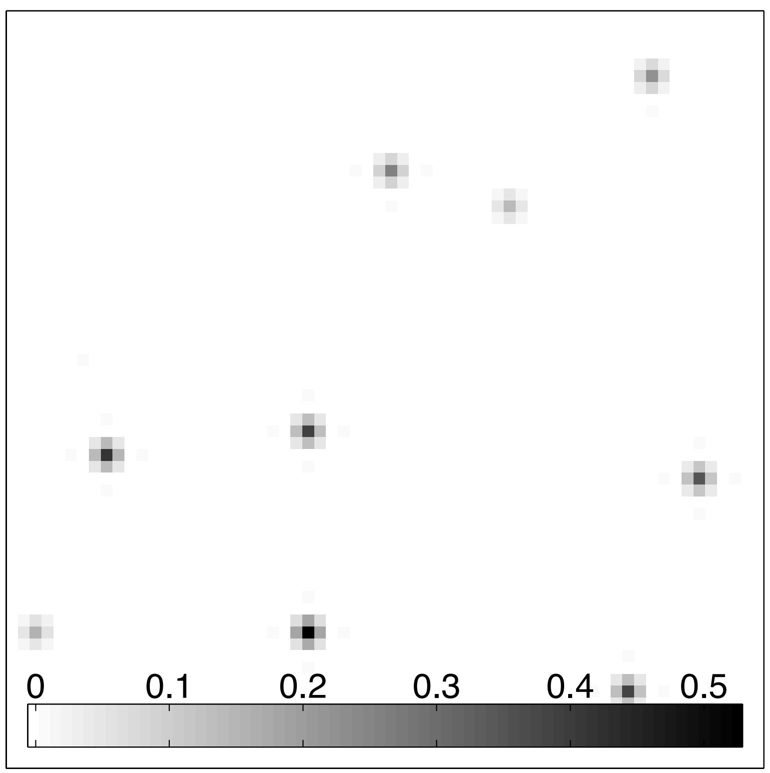
$$\boldsymbol{y} = \Theta^{(w)} \boldsymbol{\alpha} + \boldsymbol{n}$$

(30 simulations per size and M)

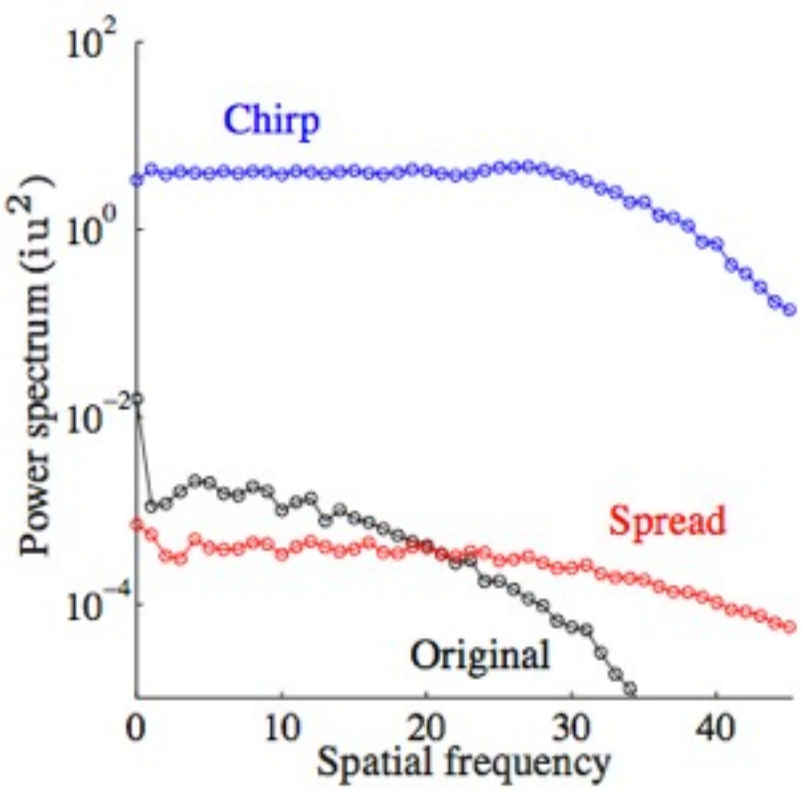
Reconstruction results

* Spread spectrum universality...

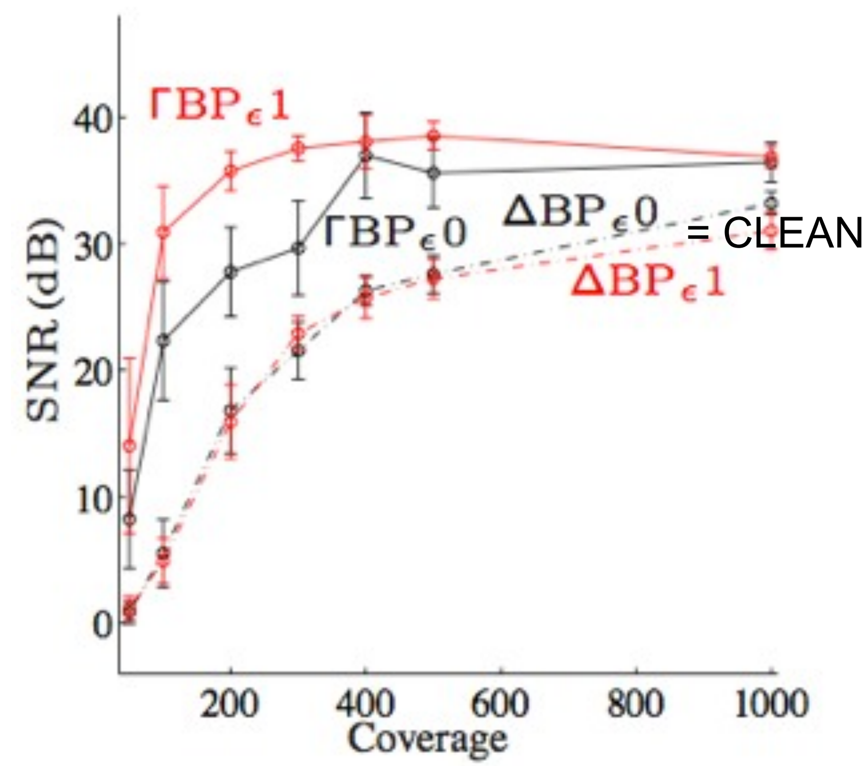
Original signal $A\boldsymbol{x}$



Spread spectrum



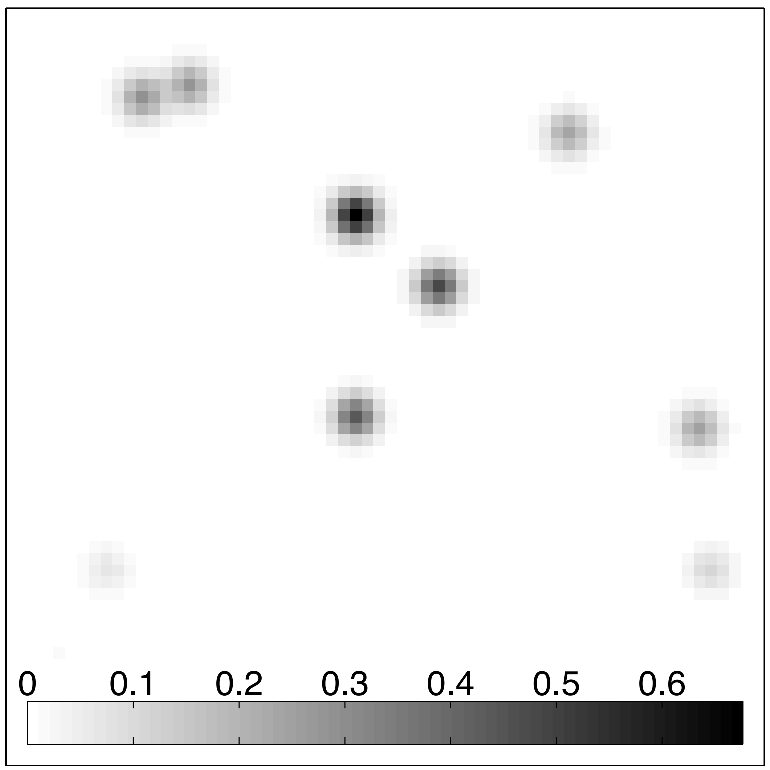
BP$_\epsilon$ reconstructions



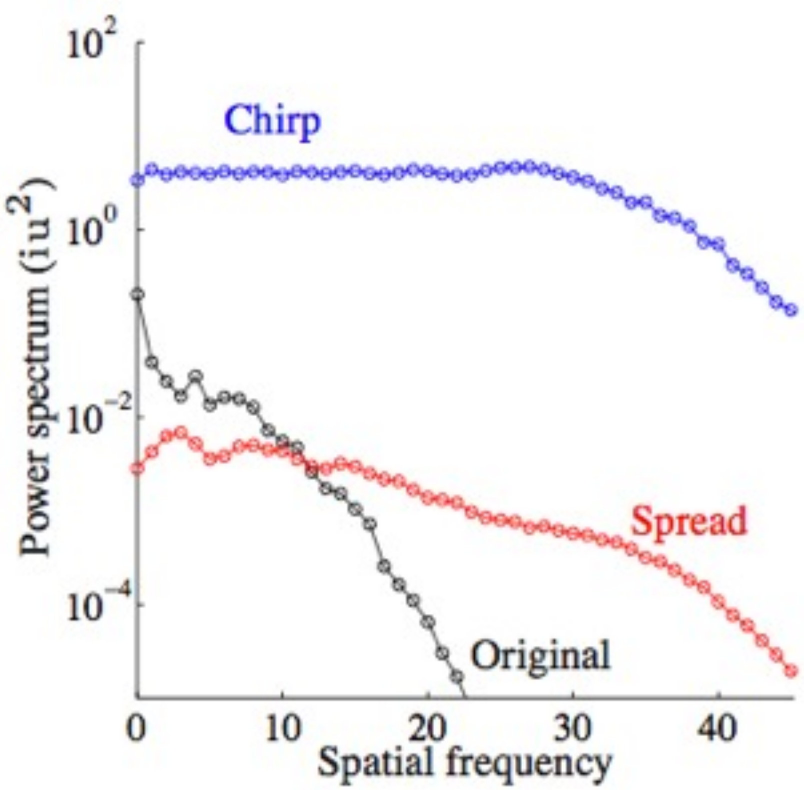
Reconstruction results

* Spread spectrum universality...

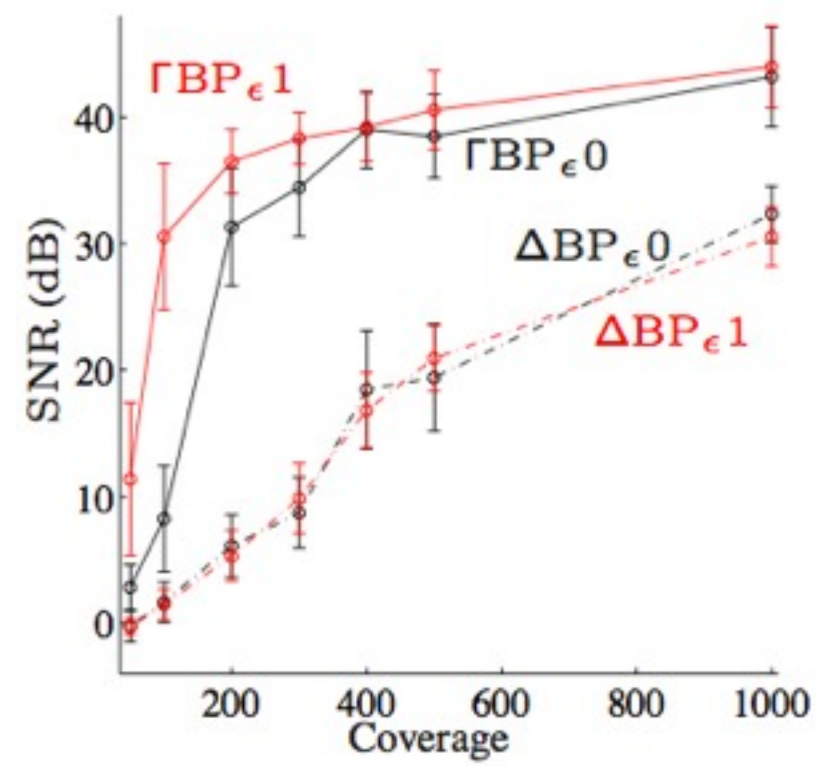
Original signal $\mathsf{A}\boldsymbol{x}$



Spread spectrum



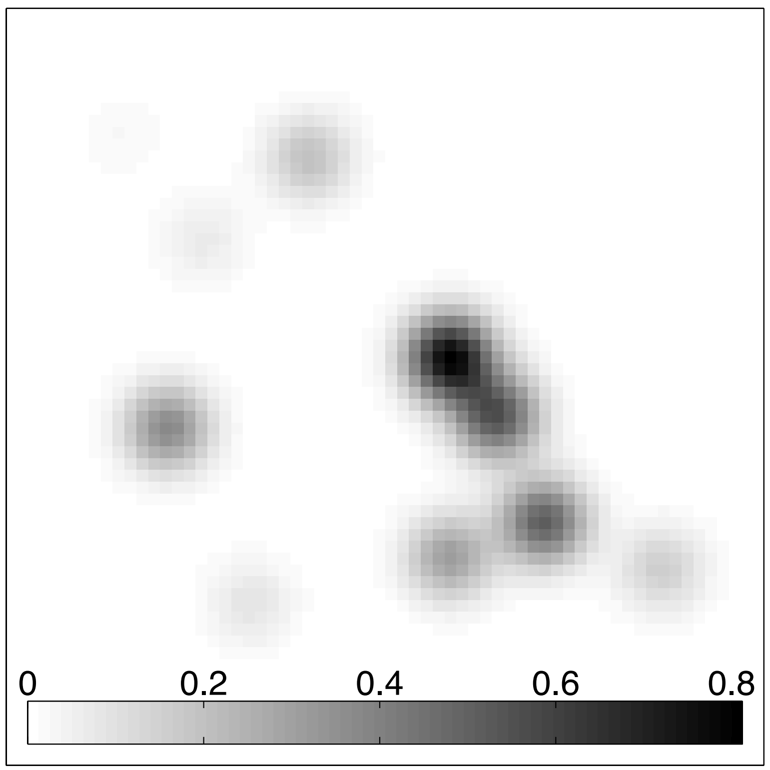
BP_ϵ reconstructions



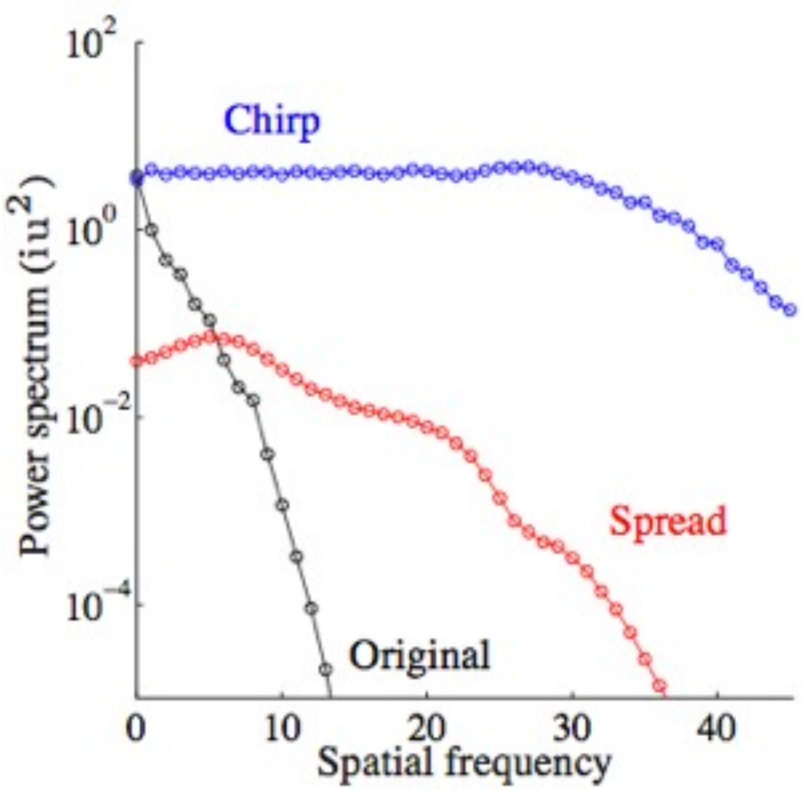
Reconstruction results

* Spread spectrum universality...

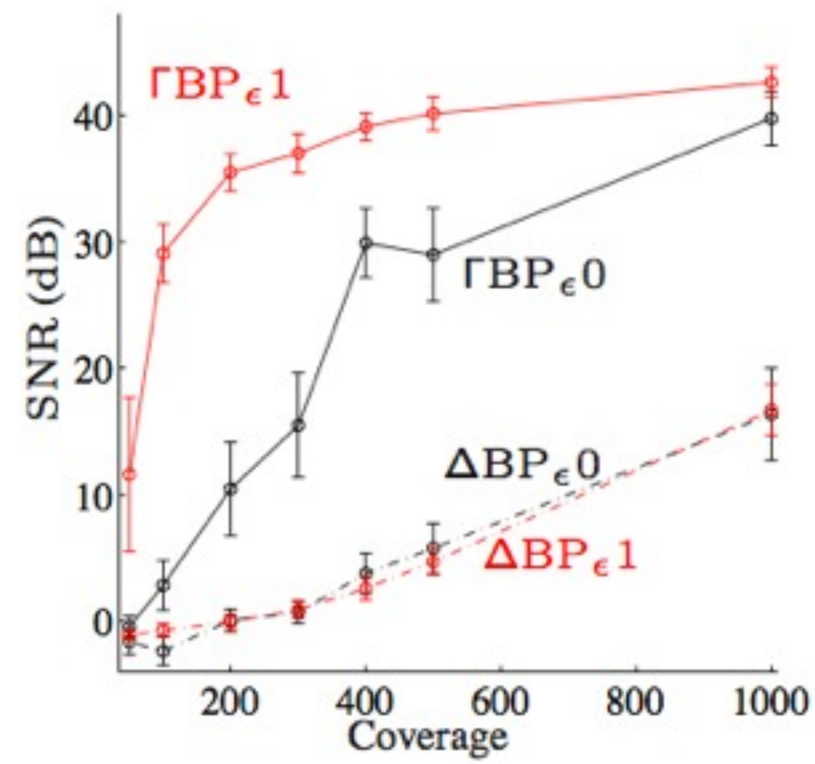
Original signal $\mathsf{A}\boldsymbol{x}$



Spread spectrum



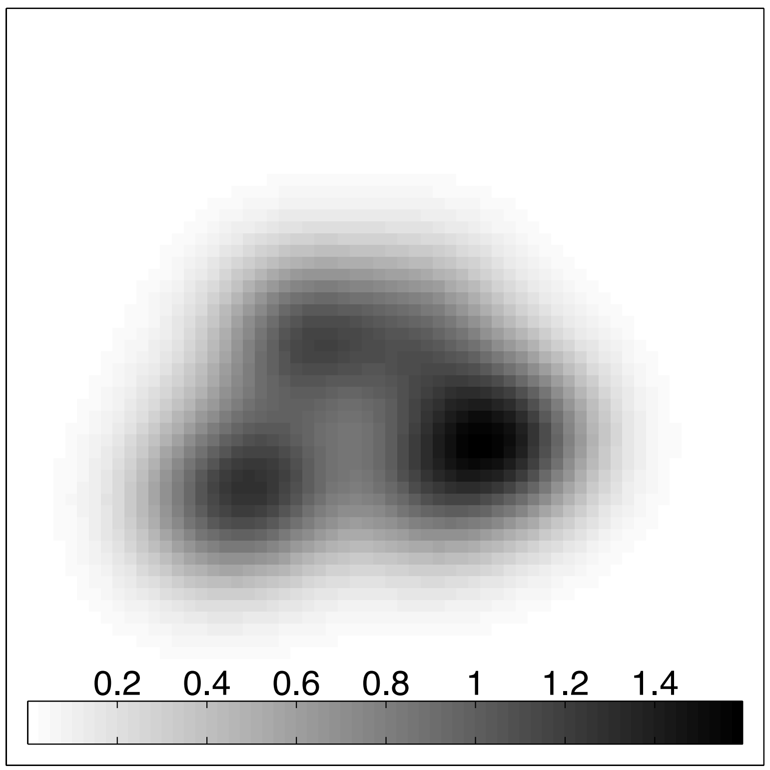
BP_ϵ reconstructions



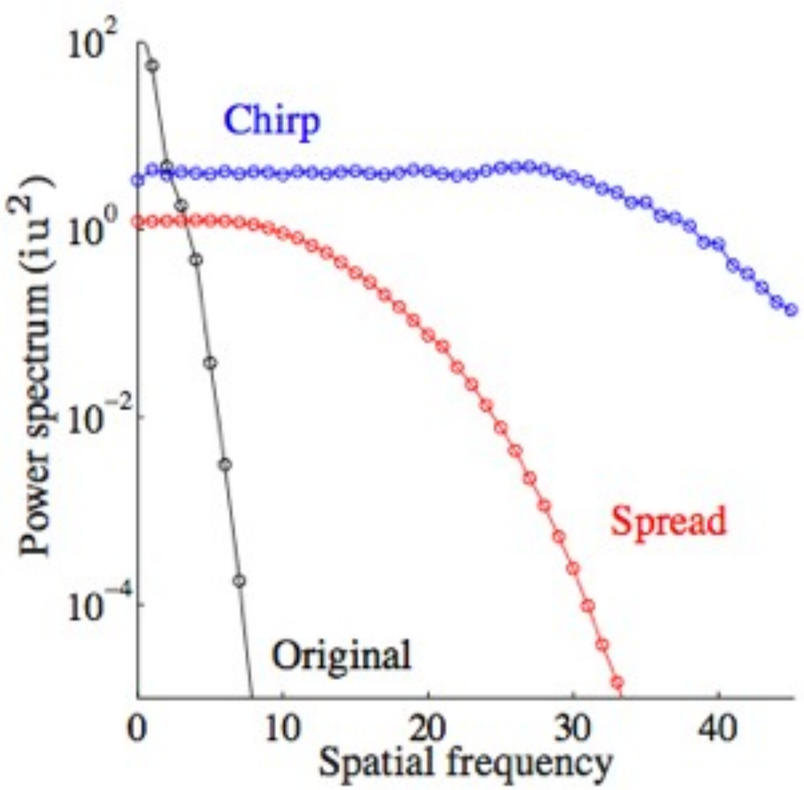
Reconstruction results

* Spread spectrum universality...

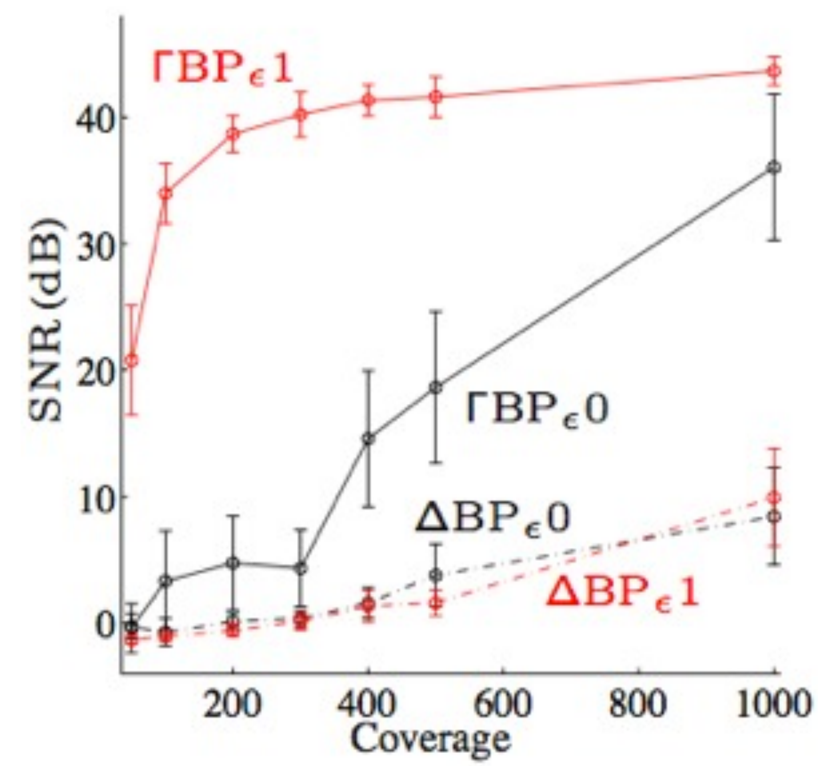
Original signal $A\boldsymbol{x}$



Spread spectrum



BP$_\epsilon$ reconstructions



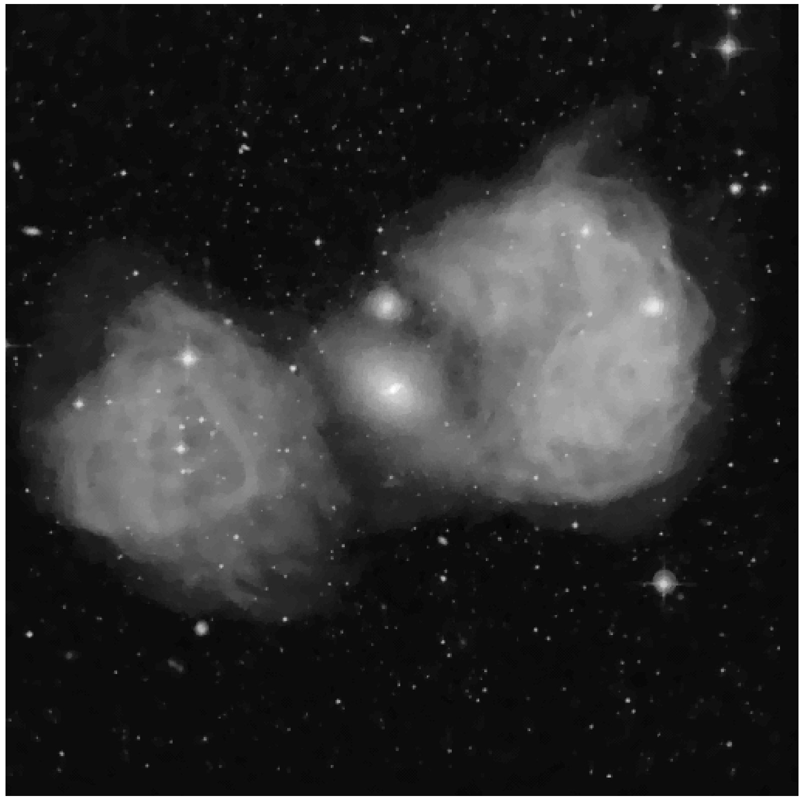
... Unveiling the mystery

* Reconstruction of the Fornax A radio emission around the elliptical galaxy NGC 1316.

Image: $N = 512 \times 512$

Simulated acquisition: 30 dB noise, **modulation, random coverage of 10%**

Reconstruction: TV_ϵ minimization problem



$\mathrm{SNR} = 21.8\,\mathrm{dB}$

Conclusions

Take-home message

Compressed sensing approaches may lead to a drastic enhancement of the image reconstruction quality for radio interferometry...
by acting both at the acquisition and reconstruction levels.

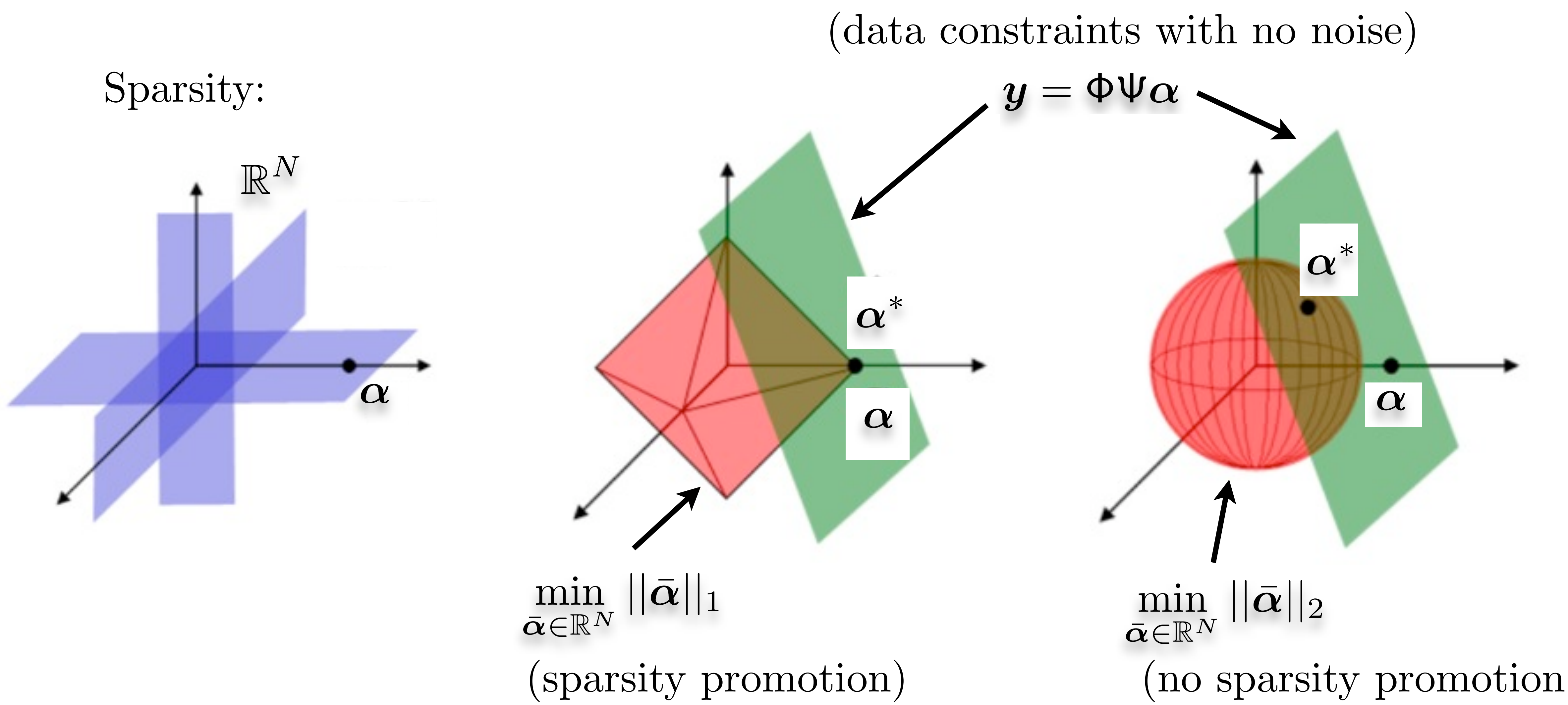
As an example: alter the acquisition to
spread the spectrum and optimize measurement incoherence... in practice, consider optimizing interferometer design to get large w on small fields of view, or consider large fields of view on the sphere (SKA, ...).

Extra-s

Why BP for sparsity?

* l_1-norm minimization promotes sparsity on the contrary of l_2-norm minimization...

- Geometrical argument...



Images: courtesy R. Baraniuk

Why BP for sparsity?

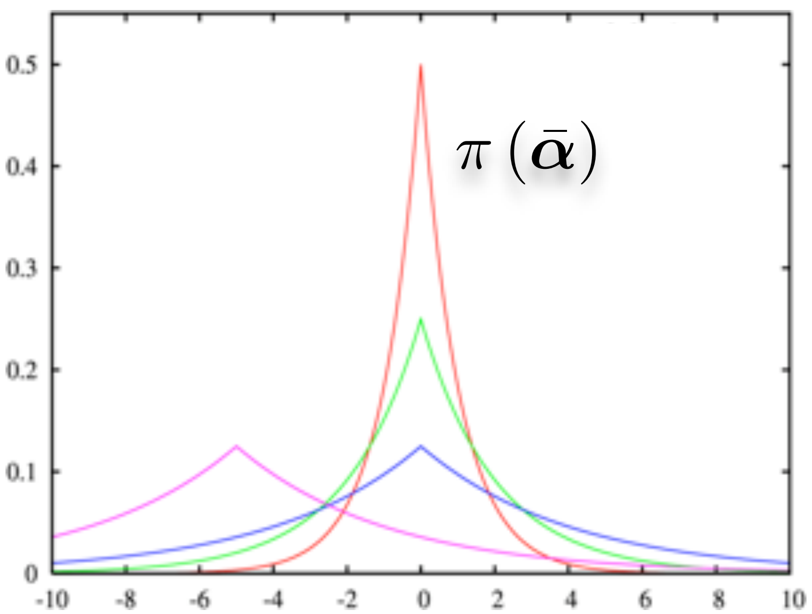
* l_1-norm minimization promotes sparsity on the contrary of l_2-norm minimization...

❖ Bayesian argument...

The constrained BP problem $\min_{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^N} ||\bar{\boldsymbol{\alpha}}||_1$ subject to $||\boldsymbol{y} - \Theta\bar{\boldsymbol{\alpha}}||_2 \leq \epsilon$

is essentially equivalent to the MAP estimation $\min_{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^N} [||\boldsymbol{y} - \Phi\Psi\bar{\boldsymbol{\alpha}}||_2^2 + \lambda||\bar{\boldsymbol{\alpha}}||_1]$

for a Laplacian prior $\pi(\bar{\boldsymbol{\alpha}}) \propto \exp[-\lambda' ||\bar{\boldsymbol{\alpha}}||_1]$



Highly peaked,
heavily tailed,
i.e. sparsity promoting

Amending standard conditions

* In practice the effect of the modulation may often be negligible...

❖ On small fields of view $L \ll 1$, the chirp rate is generally small and the spread spectrum negligible. Indeed, for a signal with band limit

$$B \simeq (u_{\max}, v_{\max}),$$

a significant modulation requires a strong alignment of baseline in the pointing direction:

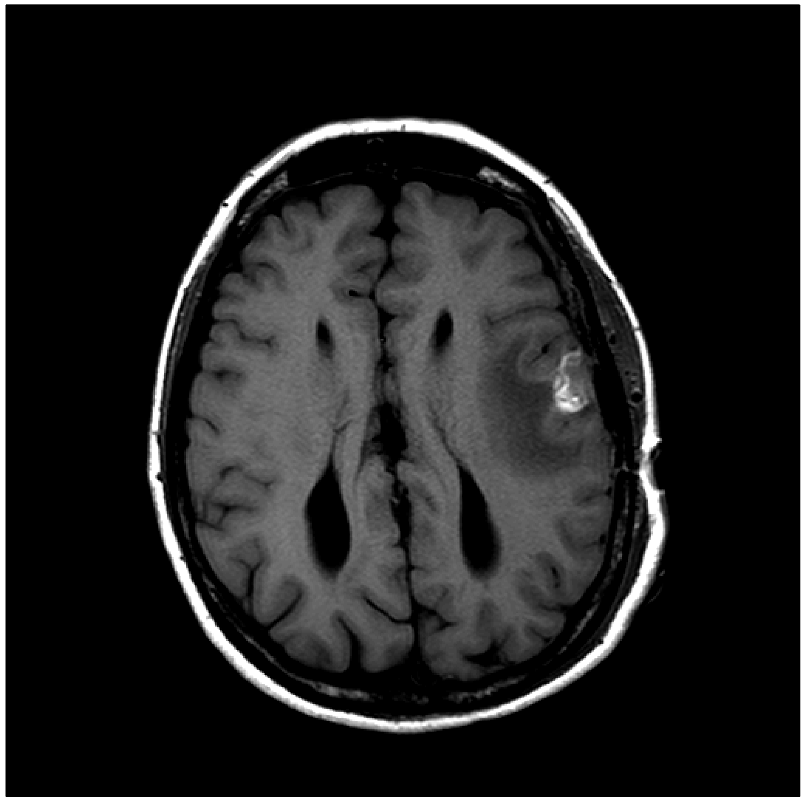
$$wL^{1/2}/2 \simeq B \text{ i.e. } w \gg (u_{\max}, v_{\max}).$$

❖ This sub-optimality could be circumvented by considering large fields of view on the sphere, or if interferometers could be designed to optimize the baseline alignment!

Another mystery...

* The magnitude of a real human brain image acquisition.

Original image: $N = 512 \times 512$



Courtesy OsiriX DICOM

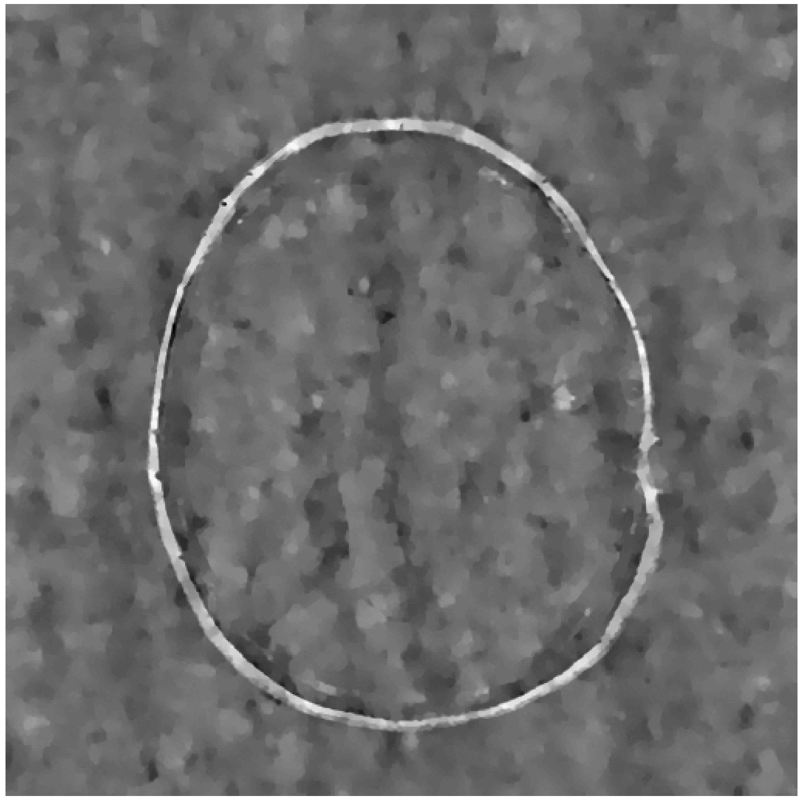
...

* Reconstruction of the human brain image.

Image: $N = 512 \times 512$

Simulated acquisition: 30 dB noise, **no modulation, acceleration by a factor 10**

Reconstruction: TV_ϵ minimization problem



$\mathrm{SNR} = 1.69\,\mathrm{dB}$

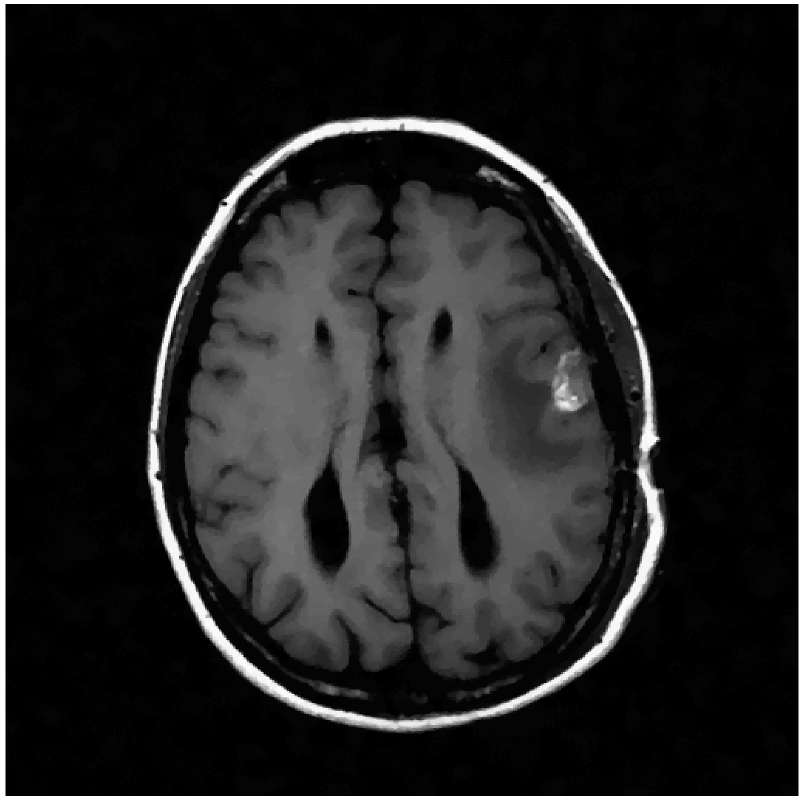
... unveiled

* Reconstruction of the human brain image.

Image: $N = 512 \times 512$

Simulated acquisition: 30 dB noise, **modulation, acceleration by a factor 10**

Reconstruction: TV_ϵ minimization problem



$\mathrm{SNR} = 23.2\,\mathrm{dB}$