# Fundamentals of Interferometry

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# Outline

- What is an interferometer?
- Basic theory
- Interlude: Fourier transforms for birdwatchers
- Review of assumptions and complications
- Interferometers and arrays: key technical issues

# Why Interferometry?

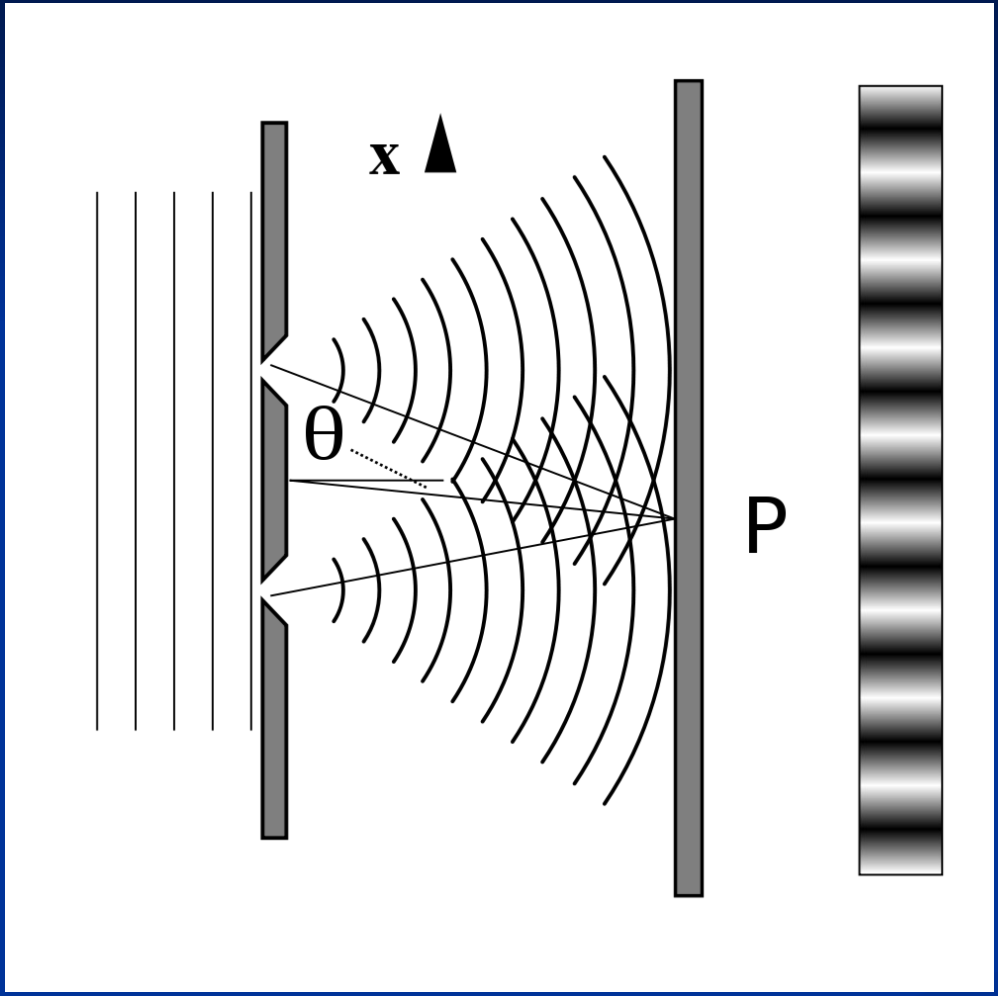
Diffraction limit for a single-dish radio telescope ~λ/D radians

Maximum aperture
D ~ 300m (Arecibo)
λ/D ~ 40 arcsec at 5 GHz

For steerable telescopes
D ~100m (Effelsberg)

Solution: interferometry. Used at optical wavelengths in the early 20$^{th}$ century by Michelson and at radio wavelengths since 1945. Resolution is now ~λ/d radians, where d is the separation of the interferometer elements - potentially d > Earth diameter. But how does this really work?

# Young's slit experiment

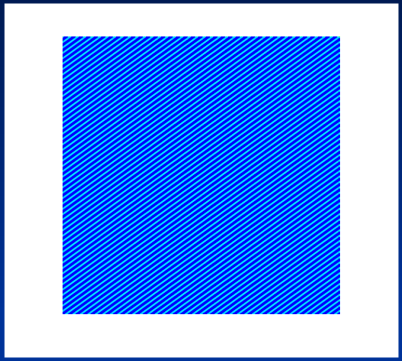
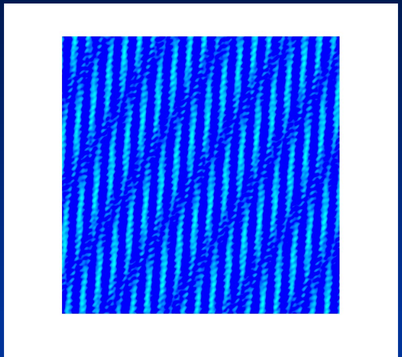
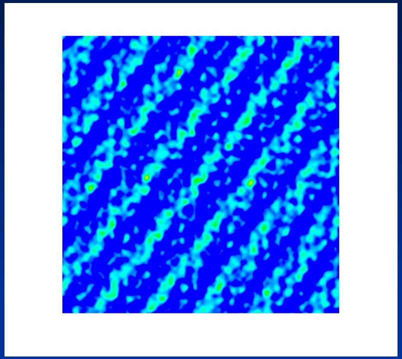
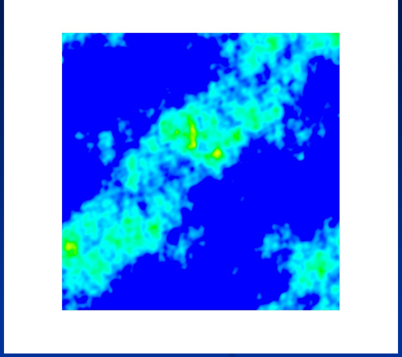
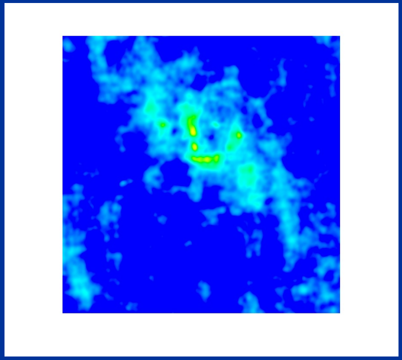
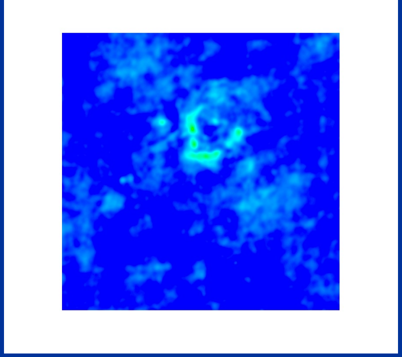
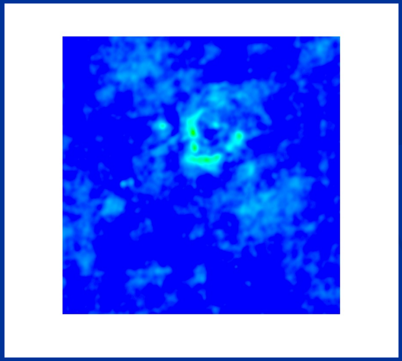
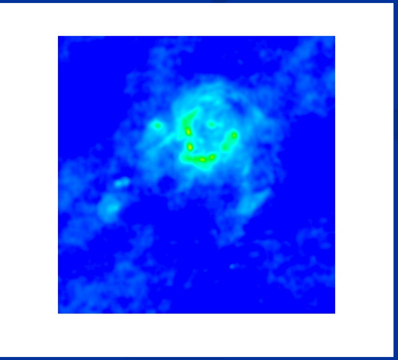



Angular spacing of fringes = λ/d

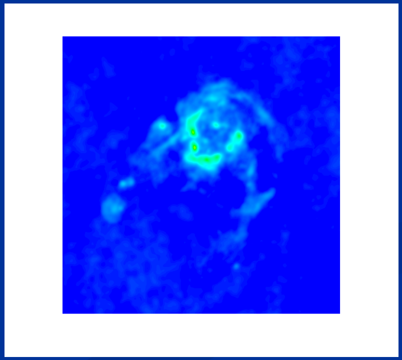
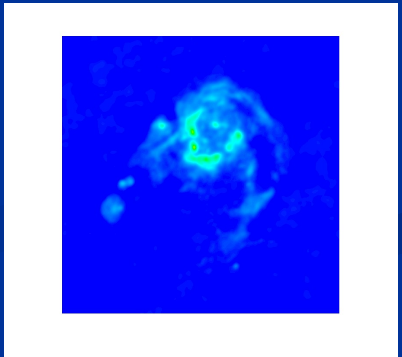
Familiar from optics

Essentially the way that astronomical interferometers operate in the optical and infrared ("direct detection")

# Build up an image from many slits

Number of apertures increases from frame to frame: 2, 4, 8, ... 1024

# But this is not how radio interferometers work in practice....

- The two techniques are closely related, and thinking of an image as built up of sinusoidal fringes from many pairs of apertures is intuitively very useful.
- But radio interferometers collect radiation (antenna), turn it into a digital signal (receiver) and generate the interference pattern in a special-purpose computer (correlator).
- How does this work?
- In order to understand the process and its assumptions, I find it simplest to start with the concept of the mutual coherence of a radio signal received from the same object at two different places.
- Many current developments involve the simplifying assumptions, so I will try to state these clearly (and return to them later).

# The ideal interferometer (1)

- An astrophysical source at location **R** causes a time-variable electric field **E**(**R**,t). An electromagnetic wave propagates to us at point **r**.
- Express the field as a Fourier series in which the only time-varying functions are complex exponentials. We are interested only in the (complex) coefficients of this series, $\mathbf{E}_\nu(\mathbf{R})$.

$$\mathbf{E}(\mathbf{R},t) = \int \mathbf{E}_\nu(\mathbf{R}) \exp(2\pi i \nu t) d\nu$$

- Simplification 1: monochromatic radiation

$E_\nu(\mathbf{r}) = \iiint P_\nu(\mathbf{R},\mathbf{r})\, E_\nu(\mathbf{R})\, dx\, dy\, dz$ where $P_\nu(\mathbf{R},\mathbf{r})$ is the propagator

- Simplification 2: scalar field (ignore polarization)
- Simplification 3: sources are all far away
- Therefore, equivalent to having all sources at a fixed distance |**R**| - no depth information.

# The ideal interferometer (2)

- Simplification 4: space between us and the sources is empty
- In this case, the propagator is quite simple (Huygens' Principle):

$$E_\nu(\mathbf{r}) = \int E_\nu(\mathbf{R})\{\exp[2\pi i|\mathbf{R}-\mathbf{r}|/c]/|\mathbf{R}-\mathbf{r}|\}\, dA$$

(dA is the element of area at distance |**R**|)

- What we can measure is the correlation of the field at two different observing locations. This is

$$C_\nu(\mathbf{r}_1,\mathbf{r}_2) = \langle E_\nu(\mathbf{r}_1)E^*_\nu(\mathbf{r}_2)\rangle$$

where <> denotes an expectation value and * means complex conjugation.

- Simplification 5: radiation from astronomical objects is not spatially coherent ('random noise').

$$\langle E_\nu(\mathbf{R}_1)E^*_\nu(\mathbf{R}_2)\rangle = 0 \text{ unless } \mathbf{R}_1 = \mathbf{R}_2$$

# The ideal interferometer (3)

- Now write $\mathbf{s} = \mathbf{R}/|\mathbf{R}|$ and $I_\nu(\mathbf{s}) = |\mathbf{R}|^2 \langle|E_\nu(\mathbf{s})|^2\rangle$ (the observed intensity). Using the approximation of large distance to the source again:

$$C_\nu(\mathbf{r}_1,\mathbf{r}_2) = \int I_\nu(\mathbf{s}) \exp\left[-2\pi i\nu\mathbf{s}.(\mathbf{r}_1-\mathbf{r}_2)/c\right] d\Omega$$

- $C_\nu(\mathbf{r}_1,\mathbf{r}_2)$, the spatial coherence function, depends only on separation $\mathbf{r}_1-\mathbf{r}_2$, so we can keep one point fixed and move the other around.
- It is a complex function, with real and imaginary parts, or an amplitude and phase.

An interferometer is a device for measuring the spatial coherence function

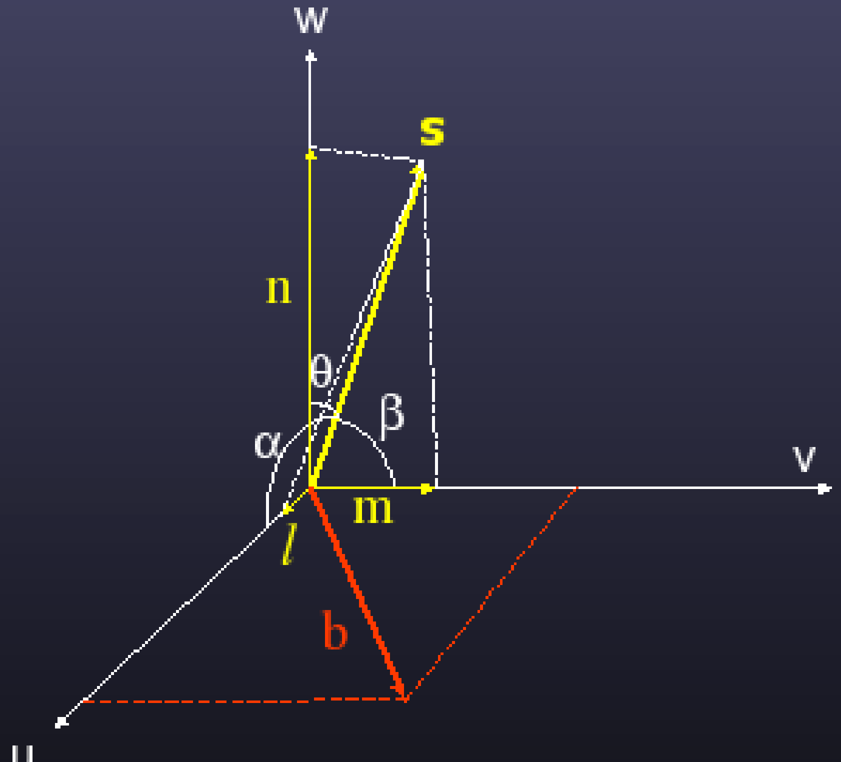
# u,v,w and direction cosines

The unit direction vector **s** is defined by its projections on the (u,v,w) axes. These components are called the **Direction Cosines**.

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$




The baseline vector **b** is specified by its coordinates (u,v,w) (measured in wavelengths). In this special case,

$$\mathbf{b} = (\lambda \mathrm{u}, \lambda \mathrm{v}, \lambda \mathrm{w}) = (\lambda \mathrm{u}, \lambda \mathrm{v}, 0)$$

# Basic Fourier Relation

- Simplification 6: receiving elements have no direction dependence
- Simplification 7A: all measurements are in the same plane, w=0.

$C(\mathbf{r}_1,\mathbf{r}_2) = V_\nu(u,v,0) = \iint I_\nu(l,m) \{\exp[-2\pi i(ul+vm)]/(1-l^2-m^2)^{1/2}\}\, dl\, dm$

- This is a Fourier transform relation between the complex visibility $V_\nu$ (the spatial coherence function with separations expressed in wavelengths) and a modified intensity $I_\nu(l,m) /(1-l^2-m^2)^{1/2}$.
- Simplification 7B: all sources are in a small region of sky
- Pick a special coordinate system such that the phase tracking centre has $s_0 = (0,0,1)$

$C(\mathbf{r}_1,\mathbf{r}_2) = \exp(-2\pi i w) V'_\nu(u,v)$, where

$V'_\nu(u,v) = \iint I_\nu(l,m) \exp[-2\pi i(ul+vm)]\, dl\, dm$

# Fourier Inversion

- In either simplified case, we can invert the Fourier transform to derive the intensity, e.g.:

$$I_\nu(l,m) = \iint V'_\nu(u,v) \exp[2\pi i(ul+vm)]\, du\, dv$$

- This is the fundamental equation of synthesis imaging.
- Interferometrists like to refer to the **(*u*,*v*)** plane. Remember that u, v (and w) are measured in wavelengths.
- Simplification 8: We have so far implicitly assumed that we can measure the visibility everywhere at a single time.

# Interlude

# Fourier transforms for birdwatchers

Some useful properties of Fourier transforms to keep in mind.

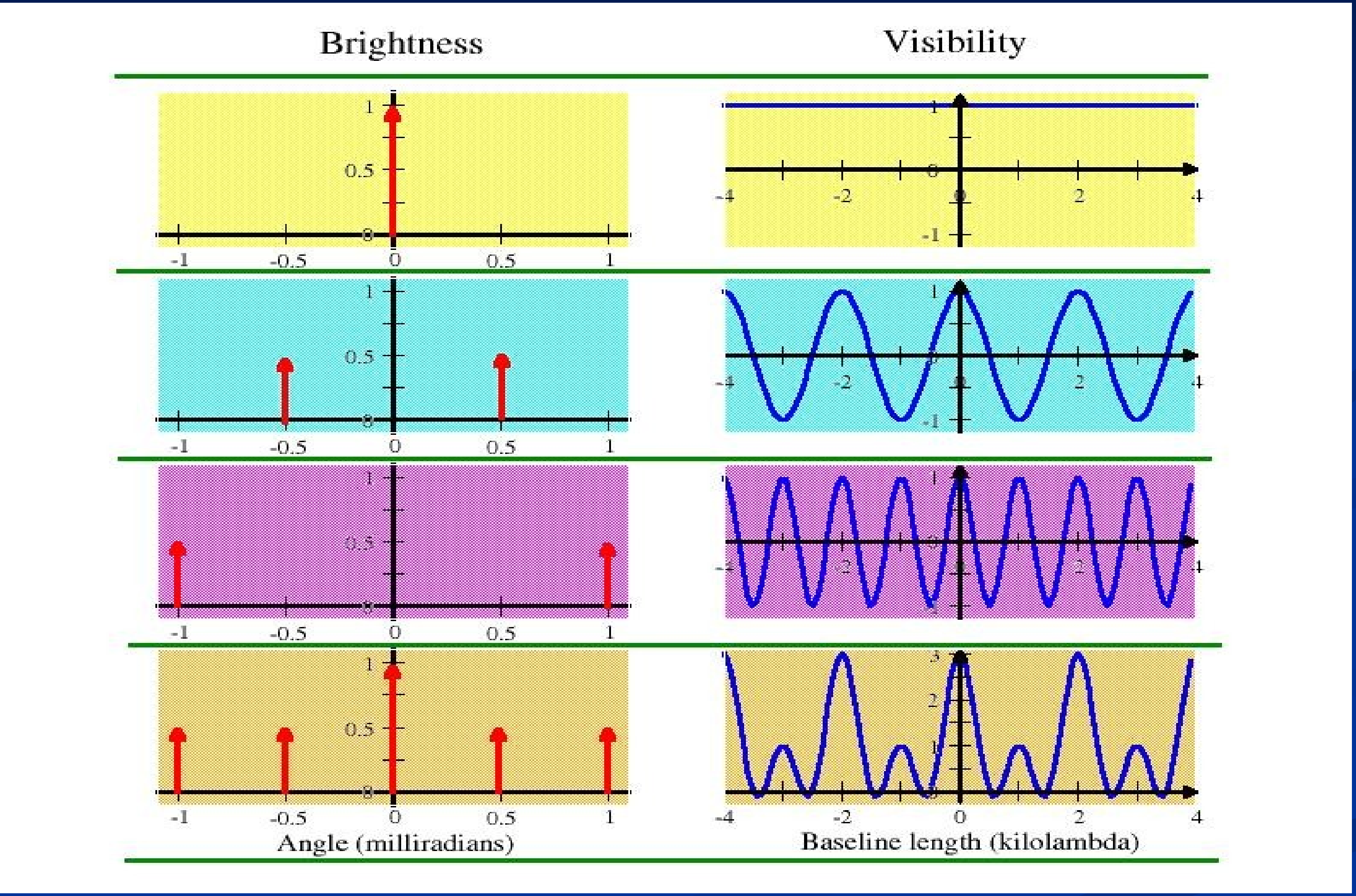
$$F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-2\pi i \nu) dt$$
$$f(t) = \int_{-\infty}^{\infty} F(s) \exp(2\pi i \nu) d\nu$$

Fourier transform pairs in one dimension

$$h(t) = \int_{-\infty}^{\infty} f(t') g(t - t') dt'$$
$$H(\nu) = F(\nu) G(\nu)$$

Convolution

# Simple 1D Fourier transform pairs

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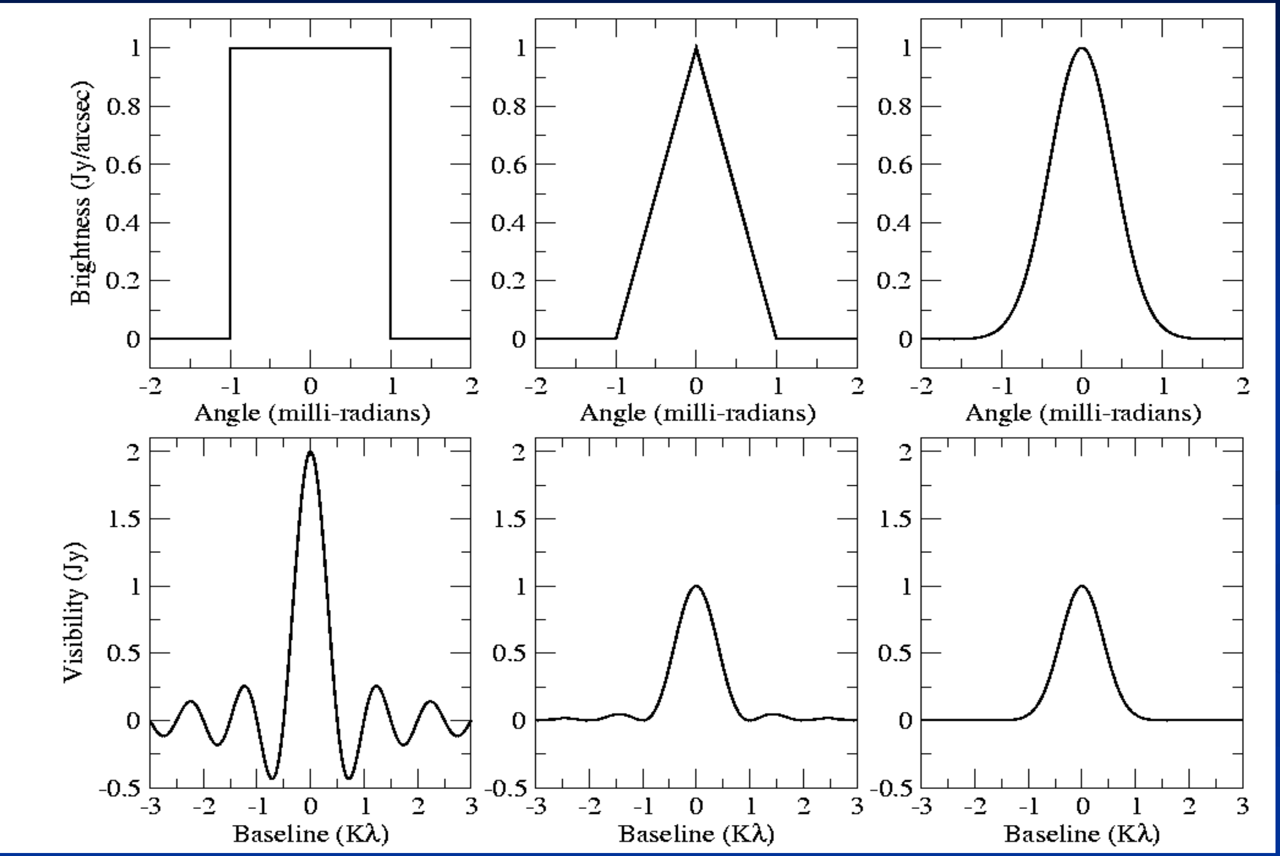
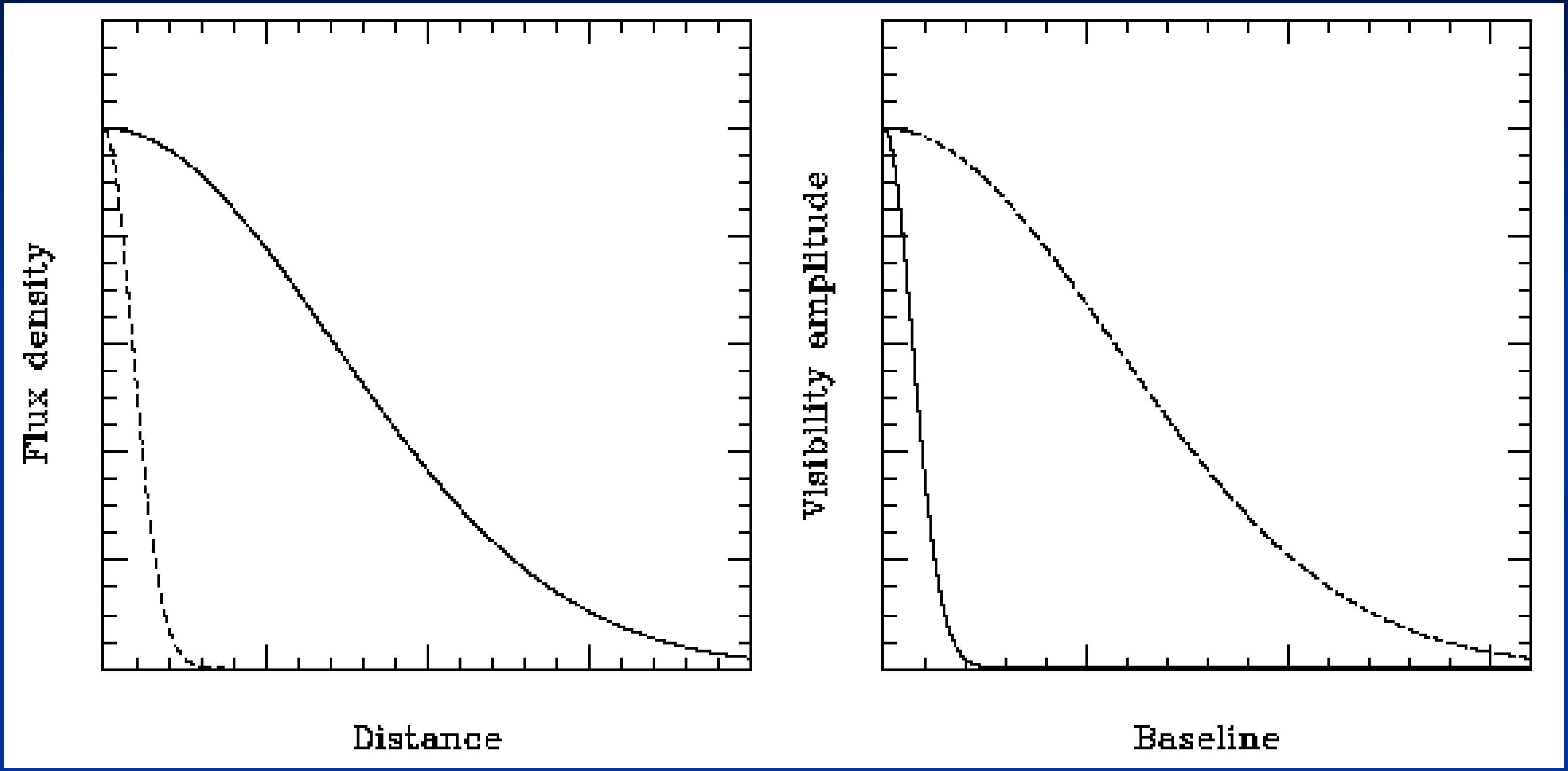
# More 1D Fourier transform pairs



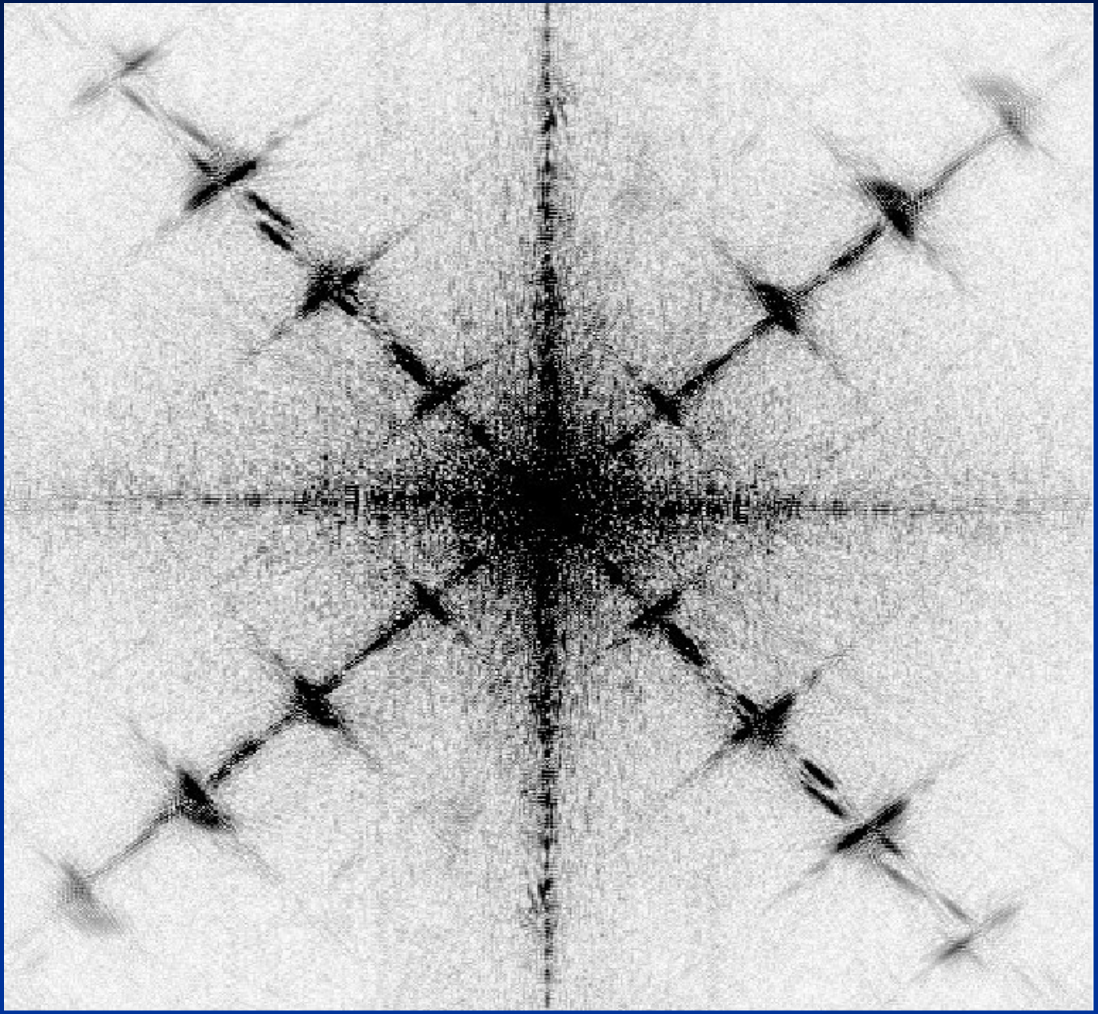

N.B.: Sharp edges in the intensity distribution lead to ripples in visibility, and vice versa.

# Fourier Transforms of Gaussians




The Fourier transform of a Gaussian function is another Gaussian
FWHM on sky is inversely proportional to FWHM in spatial frequency: fat objects have thin Fourier transforms and vice versa.

# Guess the image competition

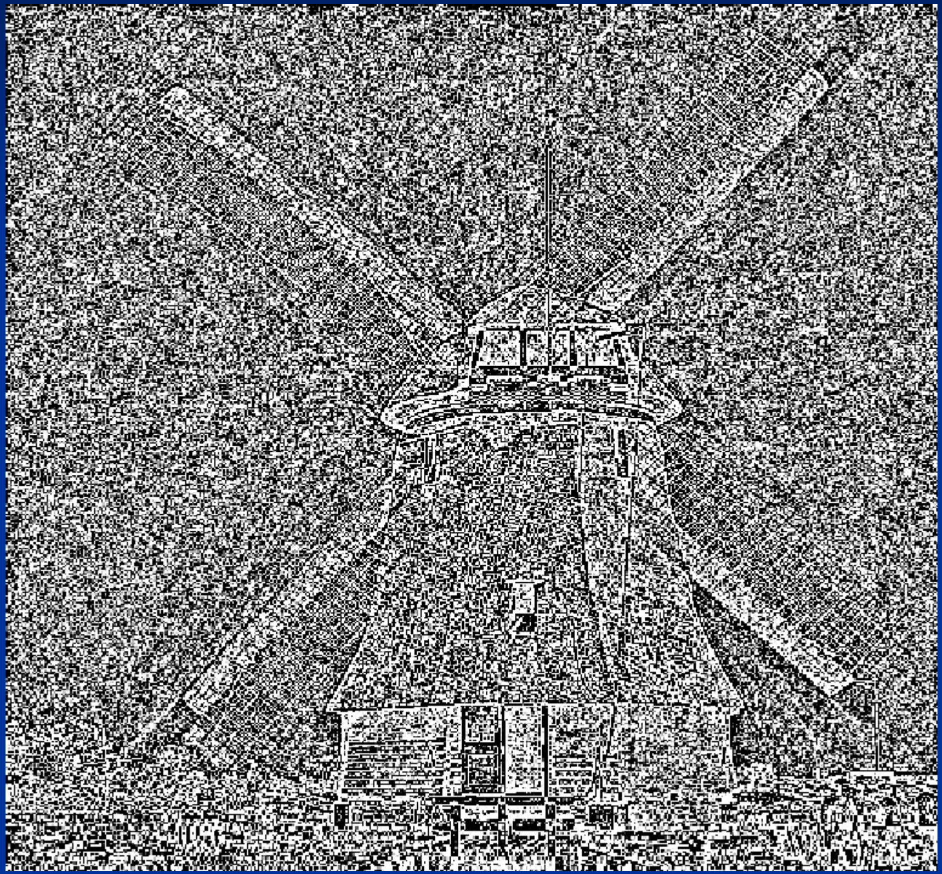


This is the amplitude of the Fourier transform of an image of a well-known object.

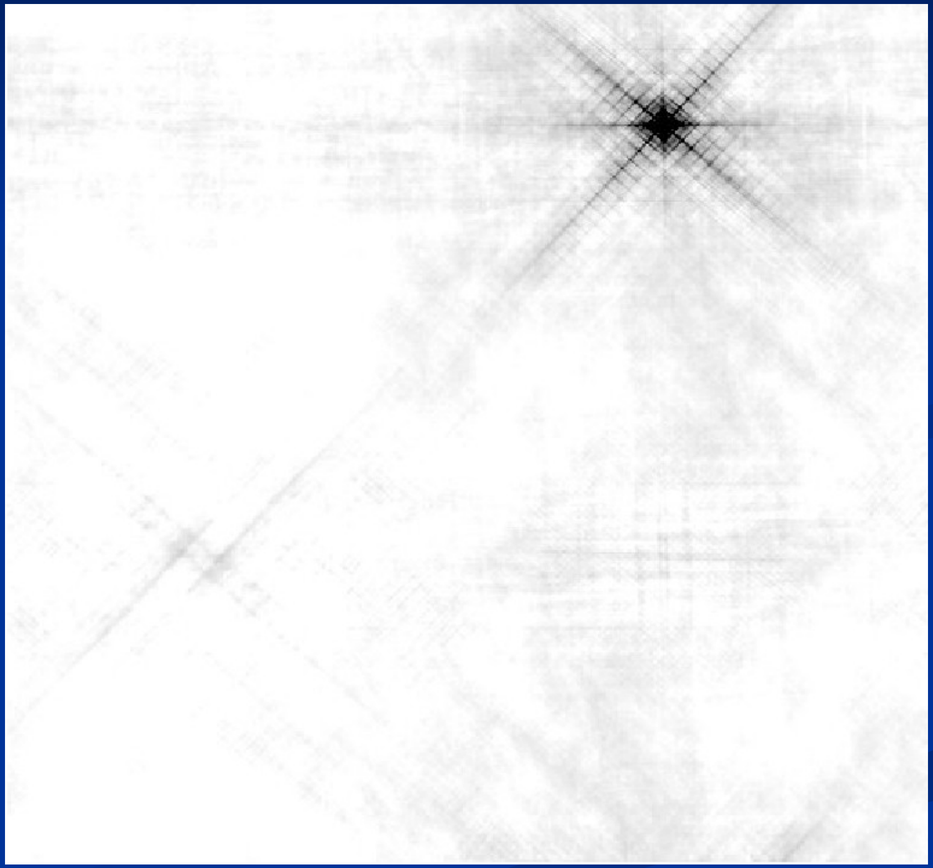
Can you:
- Say something about its size, shape and orientation?
- Deduce anything about its fine-scale structure?

# The answer

# You need phase and amplitude



Phase only



Amplitude only

# Back to Theory

# Simplification 1

- Radiation is monochromatic
- We are interested in observing wide bands, both for spectroscopy (e.g. HI, molecular lines) and for extra sensitivity for continuum imaging, so we have to get round this restriction.
- In fact, we can easily divide the band into multiple **spectral channels** (details later)
- There are imaging restrictions only if the individual channels are too wide for the field size - see imaging lectures.
  - This effect, **bandwidth smearing**, restricts the usable field of view.
  - The angular extent is roughly $(\Delta\nu/\nu_0)(l^2+m^2)^{1/2}$
  - Much less of an issue for modern correlators, which have many more frequency channels per unit frequency.

# Simplifications 2 and 3

- Treat the radiation field as a scalar quantity.
- The field is a vector, and we are interested in both components (i.e. its polarization).
- In fact this makes no difference to the analysis as long as we measure two states of polarization (e.g. right and left circular or crossed linear) and account for coupling between the states.
- Come back to this later.
- Sources are all a long way away
- Strictly speaking, this means in the far field of the interferometer, so that the distance is $> D^2/\lambda$, where D is the interferometer baseline.
- This is true except in the extreme case of very long baseline observations of solar-system objects.

# Simplification 4

- Radiation is not spatially coherent
- Generally true, even if the radiation mechanism is itself coherent (masers, pulsars)
- May become detectable in observations with extremely high spatial and spectral resolution.
- Coherence can be produced by scattering (since signals from the same location in a source are spatially coherent, but travel by different paths through the interstellar or interplanetary medium)

# Simplifications 5 and 6

- Space between us and the source is empty
- The receiving elements have no direction dependence
- Closely related and **not true** in general. Examples:
  - Interstellar or interplanetary scattering
  - Ionospheric and tropospheric fluctuations (which lead to path/phase and amplitude errors, sometimes seriously direction-dependent)
  - Ionospheric Faraday rotation, which changes the plane of polarization.
  - Antennas are usually designed to be highly directional
- Standard calibration deals with the case that there is no direction dependence (i.e. each antenna has an associated amplitude and phase which may be time-variable)
- Direction dependence is harder to deal with, but is becoming more important as field sizes increase.

# Primary Beam

- If the response of the antenna is direction-dependent, then we are measuring

$I_\nu(l,m)\ D_{1\nu}(l,m)D^*_{2\nu}(l,m)$ instead of $I_\nu(l,m)$ (ignore polarization for now)

- An easier case is when the antennas all have the same response

$A_\nu(l,m)\ = |D_\nu(l,m)|^2$

- In this case, $V'_\nu(u,v) = \iint A_\nu(l,m) I_\nu(l,m) \exp[-2\pi i(ul+vm)]\ dl\ dm$
- We just make the standard Fourier inversion and then divide by the **primary beam** $A_\nu(l,m)$

# Simplification 7

- (A) Antennas are in a single plane or (B) the field is small
- Not true for wide-field imaging (except for snapshots)
- Particularly vital at low frequencies
- Basic imaging equation becomes:

$$V_\nu(u,v,w) = \iint I_\nu(l,m)\,\{\exp[-2\pi i(ul+vm+(1-l^2-m^2)^{1/2}w)]/(1-l^2-m^2)^{1/2}\}\,dl\,dm$$

- No longer a 2D Fourier transform, so analysis becomes much more complicated (the “w term”)
  - Map individual small fields (“facets”) and combine later, or
  - w-projection
- See lectures imaging

# Simplification 8

- We have implicitly assumed that we can measure the visibility function everywhere at a single time.
- In fact:
  - We have a number of antennas at fixed locations on the Earth
  - The Earth rotates
  - We make many (usually) short integrations over extended periods, sometimes in separate observations
- Implies that the source does not vary
  - Allow source model to change with time in constrained way
- We effectively measure at discrete u, v (and w) positions.
  - The following assumes that integrations have infinitesimal duration. If this is not a good approximation, then there may be **time-average** smearing.

# Sampling

- In 2D, this process can be described by a sampling function S(u,v) which is a delta function where we have taken data and zero elsewhere.
- $I^D_\nu(l,m) = \iint V_\nu(u,v)\ S(u,v)\ \exp[2\pi i(ul+vm)]\ du\ dv$ is the **dirty image**, which is the Fourier transform of the **sampled** visibility data.
- Using the convolution theorem:

$$I^D_\nu(l,m) = I_\nu(l,m) \otimes B(l,m)$$

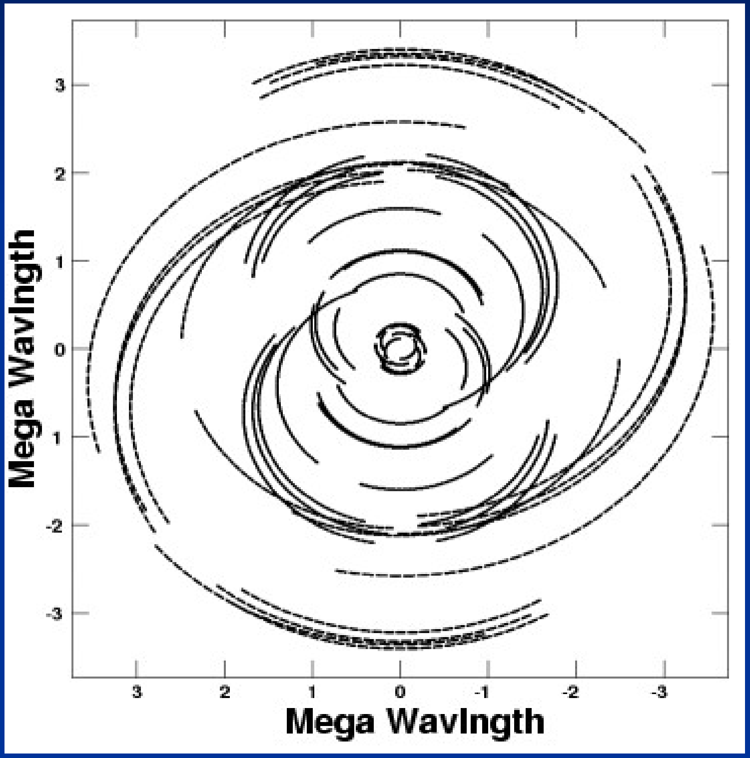
where the ⊗ denotes convolution and

$$B(l,m) = \iint S(u,v) \exp[2\pi i(ul+vm)]\ du\ dv$$

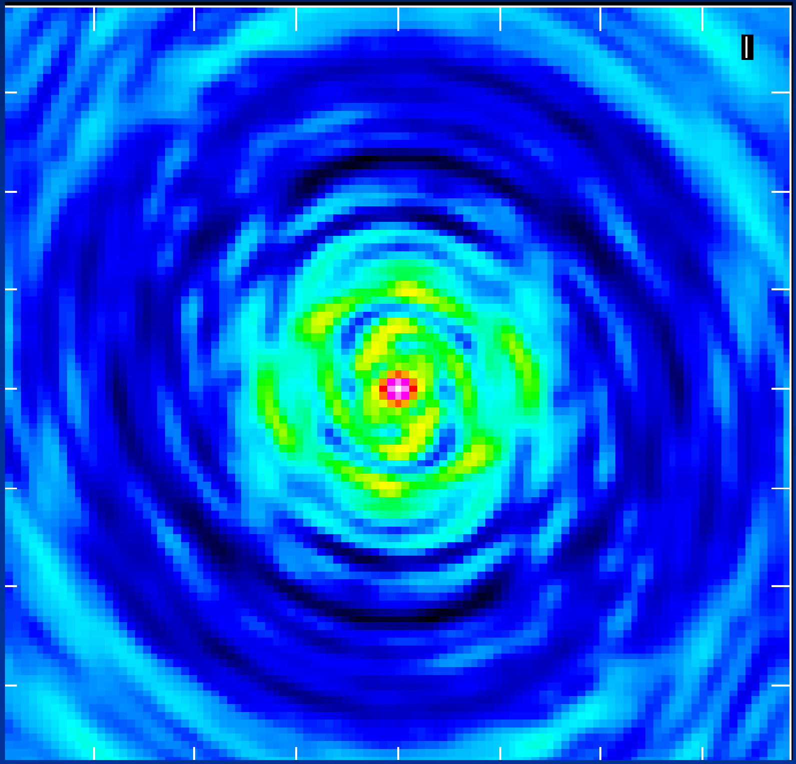
is the **dirty beam**.

- The dirty image is the convolution of the true image of the sky with the dirty beam. Working out the true image of the sky from this is **deconvolution**.

# Sampling and Imaging



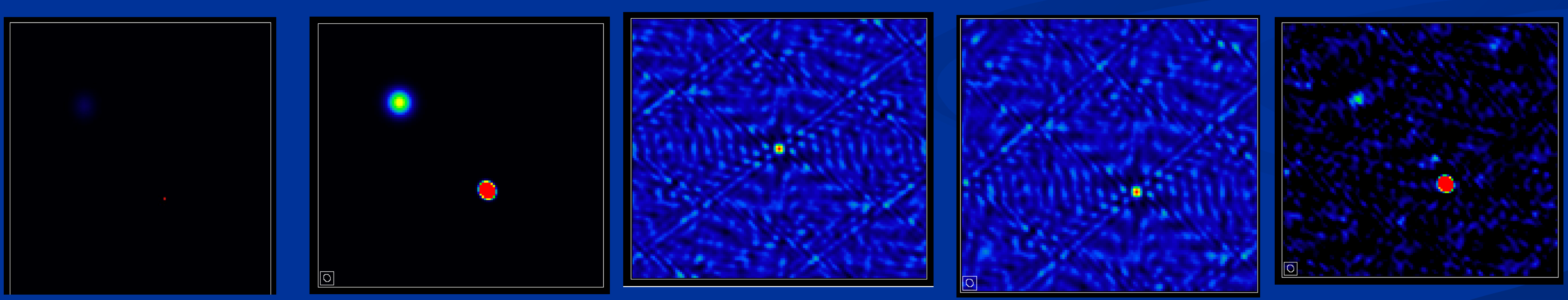

uv coverage




dirty beam

# Deconvolution

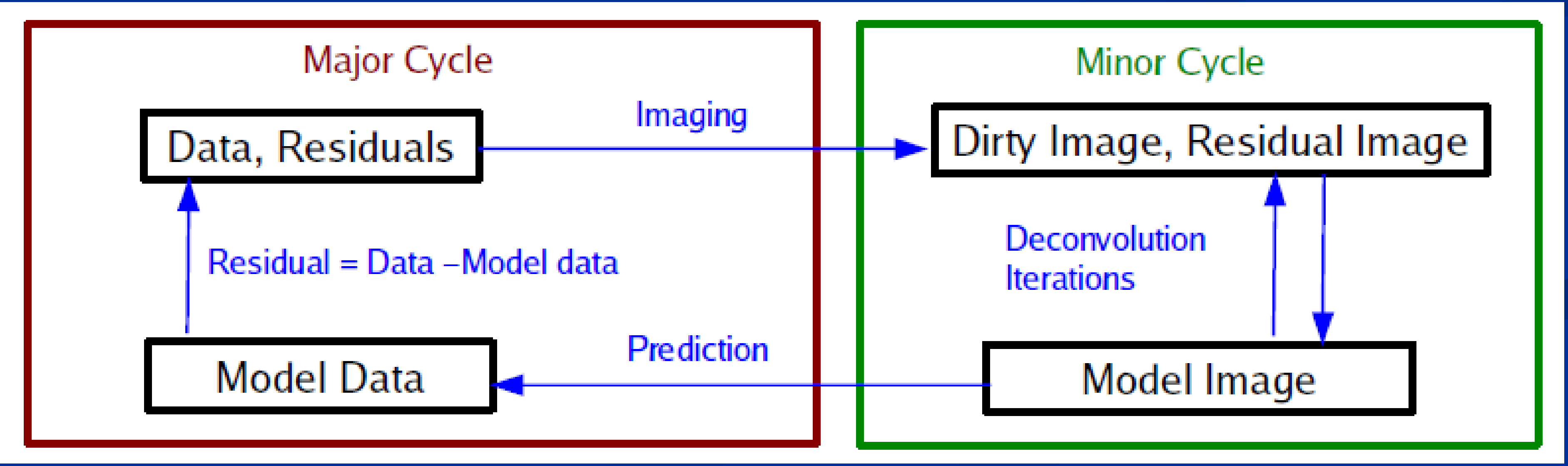
- The next stage in the imaging process is to estimate the convolution of the sky with a well-behaved **restoring beam** (usually a Gaussian function) rather than the dirty beam. This is **deconvolution**.
- Methods for this include variants of CLEAN and maximum entropy – see lecture on imaging.



Model Convolved model Dirty beam Dirty image (noise added) CLEAN image

# Gridding and FFT's

- Computationally efficient to use Fast Fourier Transform algorithm
  - Requires data to be interpolated onto $2^N \times 2^M$ grid
  - Subtle errors introduced
- More accurate (once you have a good model of the visibilities) to FT model, subtract this from the ungridded visibility data and continue working on the residual visibilities



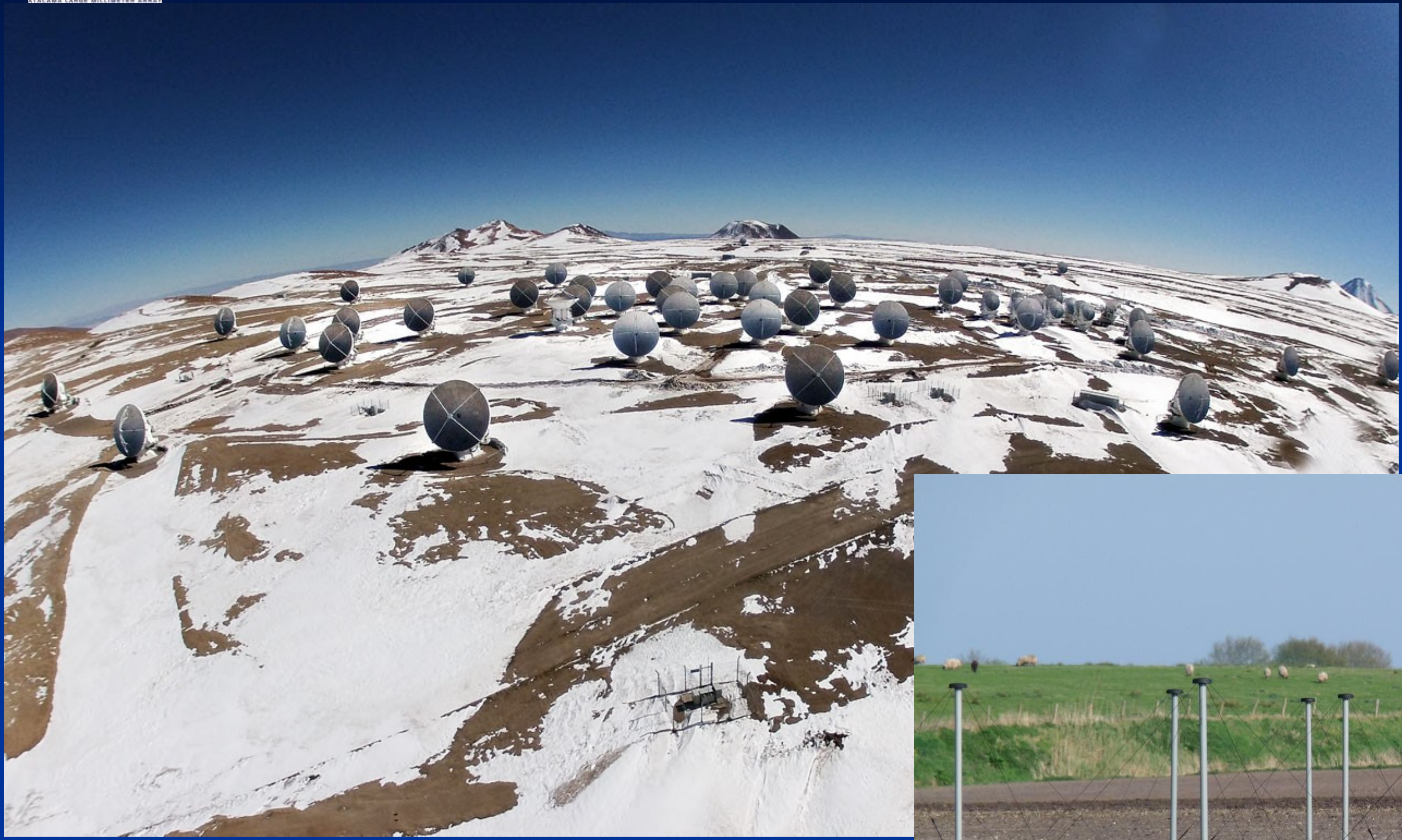
# Resolution, maximum scale and field size

- Some useful parameters:
  - Resolution /rad: ≈$\lambda/d_{max}$
  - Maximum observable scale /rad: ≈$\lambda/d_{min}$
  - Primary beam/rad: ≈$\lambda/D$
- Good coverage of the u-v plane (many antennas, Earth rotation) allows high-quality imaging.
- Some brightness distributions are in principle undetectable:
  - Uniform
  - Sinusoid with Fourier transform in an unsampled part of the u-v plane.
- Sources with all brightness on scales $>\lambda/d_{min}$ are resolved out.
- Sources with all brightess on scales $< \lambda/d_{max}$ look like points,

# Key technical issues – following the signal path

- Antennas and receivers
- Noise
- Down-conversion
- Measuring the correlations
- Spectral channels
- Measuring and describing polarization
- Calibration formalism

# Antennas



Antennas collect radiation

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# Receivers

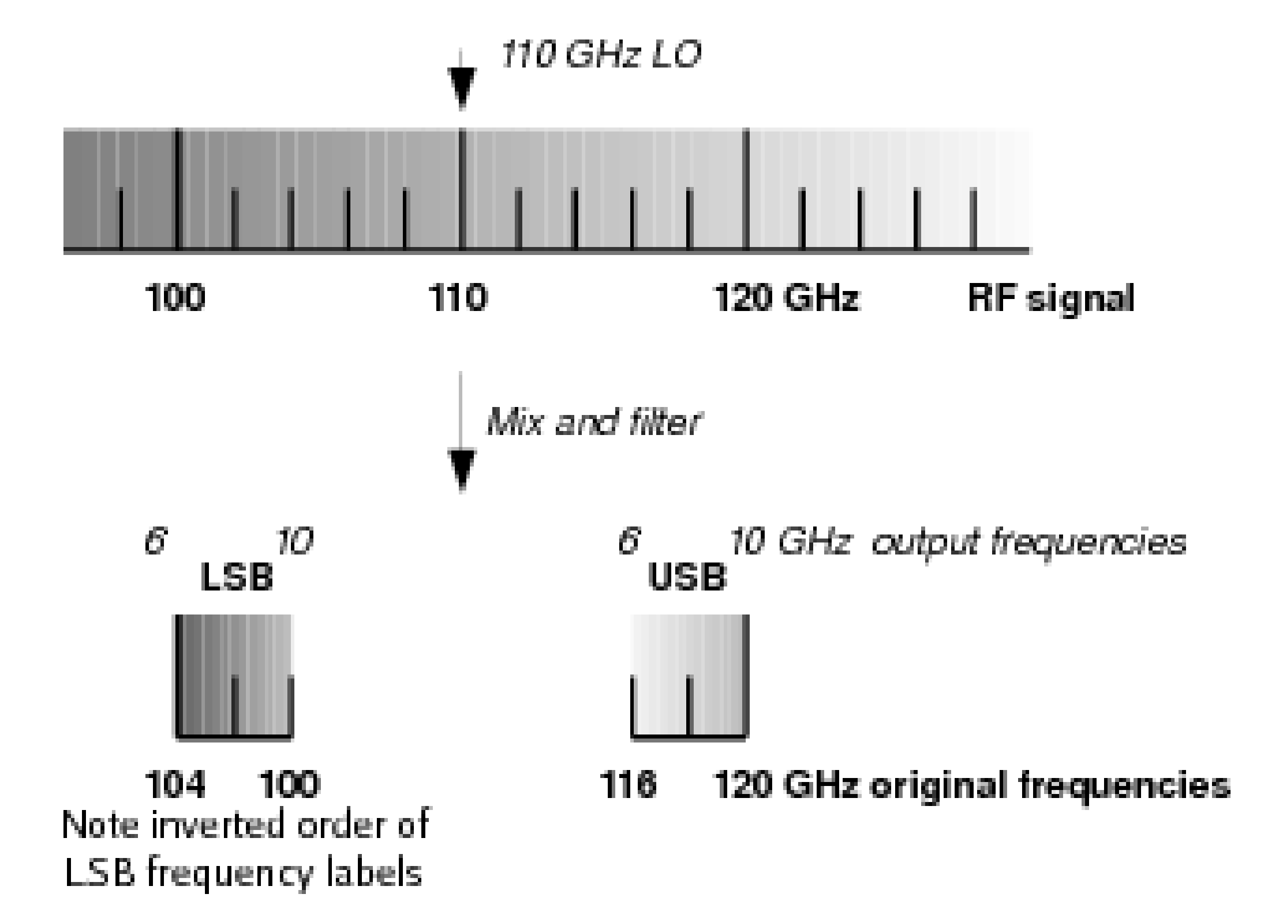
- Detect radiation
- Cryogenically cooled for low noise (except at low frequencies)
- Normally detect two polarization states
- Optionally, in various combinations:
  - Amplify RF signal,
  - mix with phase-stable local oscillator signal to make intermediate frequency (IF) $\rightarrow$ two sidebands (one or both used)
  - Additional stages of frequency conversion and/or filtering
- Digitize in coarse frequency blocks variously known as IF's, basebands, sub-bands, etc.
- Send to correlator (e.g. over optical fibre) or store (VLBI).

# Noise

- RMS noise level $S_{rms}$
  - $T_{sys}$ is the system temperature, $A_{eff}$ is the effective area of the antennas, $N_A$ is the number of antennas, $\Delta\nu$ is the bandwidth, $t_{int}$ is the integration time and k is Boltzmann's constant
- For good sensitivity, you need low $T_{sys}$ (receivers), large $A_{eff}$ (big, accurate antennas), large $N_A$ (many antennas) and, for continuum, large bandwidth $\Delta\nu$.

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A-1)t_{int}\Delta\nu}}$$

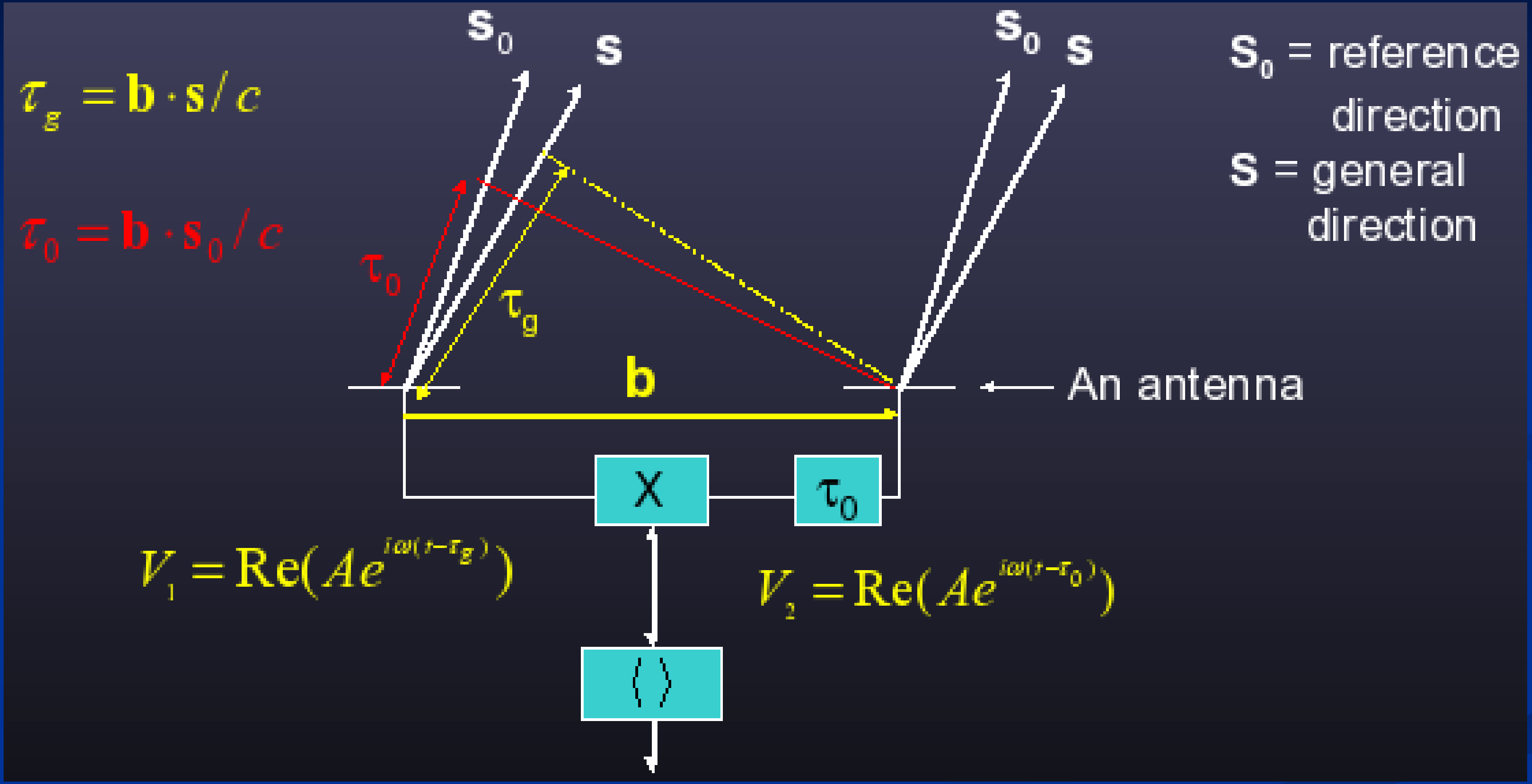
# Example: ALMA signal flow



# Delay

- An important quantity in interferometry is the time delay in arrival of a wavefront (or signal) at two different locations, or simply the delay, $\tau$.
  - Constant ("cable") delay in waveguide or electronics
  - geometrical delay
  - propagation delay through the atmosphere
  - ......
- Phase varies linearly with frequency for a constant delay
  - $\Delta\varphi = 2\pi\tau\Delta\nu$
  - Characteristic signature
  - Beware of confusion between phase and delay in discussions of narrow-band imaging
  -

# Delay tracking




Works exactly only for the delay tracking centre. Maximum averaging time is a function of angle from this direction.

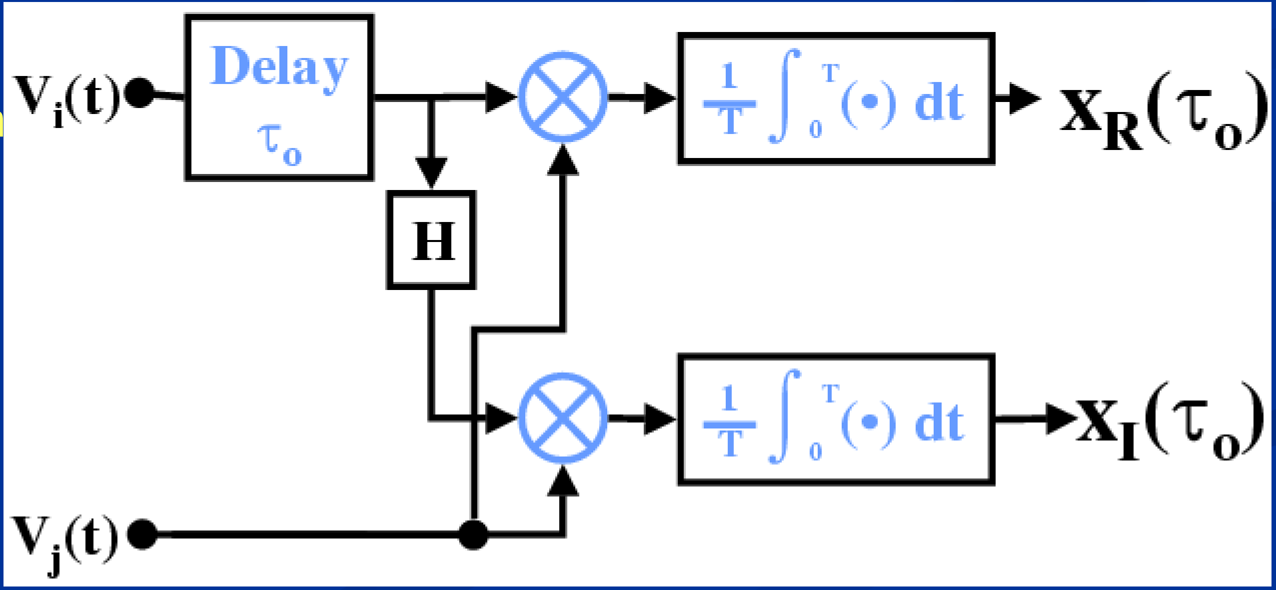
# Complex correlator

$$v_i(t) = \sin 2\pi\nu_0 t$$
$$v_j(t) = \sin(2\pi\nu_0 t + \phi)$$

$$x_{ij}(\tau) = \langle \sin 2\pi\nu_0 t) \sin(2\pi\nu_0(t+\tau)+\phi)\rangle$$
$$= x_R \cos 2\pi\nu_0(\tau - \tau_0) + x_I \sin 2\pi\nu_0(\tau-\tau_0)$$

- $x_R = x_{ij}(\tau_0)$
- $x_I = x_{ij}(\tau_0 + \Delta\tau)$, **with** $\Delta\tau = 1/(4\nu_0)$ $(\Delta\phi = 90°)$.

Signals from antennas




Real

Imaginary

# Fringe stopping

- The problem with downconversion is that the geometrical delay now has to be applied in the IF part of the signal path, whereas it should apply to the RF signal
- Correct phase is $2\pi\nu_{RF}(\tau_g - \tau_0)$
- Actual phase is $2\pi\nu_{RF}\tau_g - 2\pi\nu_{IF}\tau_0 - \varphi_{LO}$
- These are equal provided that $\varphi_{LO} = 2\pi\nu_{LO}\tau_0$
- Applying this phase change to the LO is called fringe stopping. It defines the phase tracking centre.
- There are actually three different "field centre" positions, normally but not always set to be the same: pointing, delay and phase tracking.

# Multiple spectral channels

- We make multiple channels by correlating with different values of **lag**, $\tau$. This is a delay introduced into the signal from one antenna with respect to another as in the previous slide. For each quasi-monochromatic frequency channel, a lag is equivalent to a phase shift $2\pi\tau\nu$, i.e.

$$V(u,v,\tau) = \int V(u,v,\nu)\exp(2\pi i\tau\nu)\, d\nu$$

- This is another Fourier transform relation with complementary variables $\nu$ and $\tau$, and can be inverted to extract the desired visibility as a function of frequency.
- In practice, we do this digitally, in finite frequency channels:

$$V(u,v,j\Delta\nu) = \Sigma_k V(u,v, k\Delta\tau) \exp(-2\pi ijk\Delta\nu\Delta\tau)$$

- Each spectral channel can then be imaged (and deconvolved) individually. The final product is a **data cube**, regularly gridded in two spatial and one spectral coordinate.

# Calibration

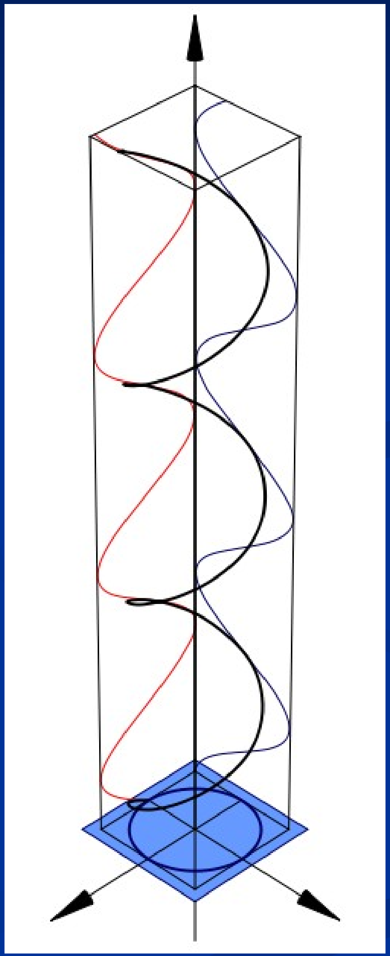
- What we want from this step of data processing is a set of perfect visibilities V(u,v,w,ν), on an absolute amplitude scale, measured for exactly known baseline vectors (u,v,w), for a set of frequencies, ν, in full polarization.
- What we have is the output from a correlator, which contains signatures from at least:
  - the Earth's atmosphere
  - antennas and optical components
  - receivers, filters, amplifiers
  - digital electronics
  - leakage between polarization states
- ... not to mention bad data (interference, mispointing antennas, dead electronics, etc.)

# Polarization

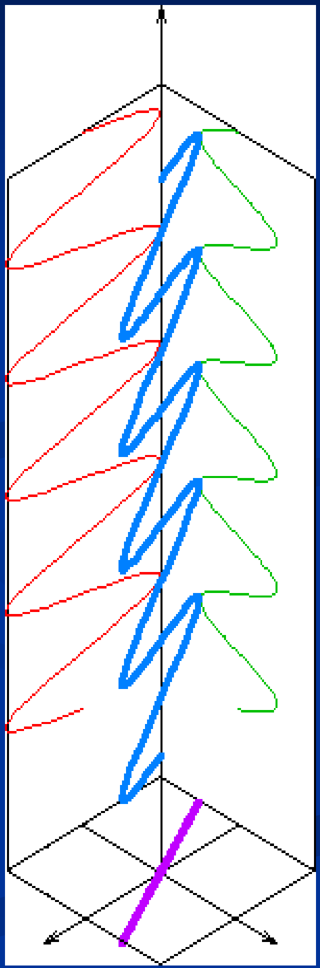
Want to image
Stokes Parameters

I (total intensity)
Q, U (linear)
V (circular)

Fractional linear polarization
$= (Q^2+U^2)^{1/2}/I$
Position angle
$=(1/2)\ \arctan(U/Q)$
Fractional circular polarization
$= |V|/I$



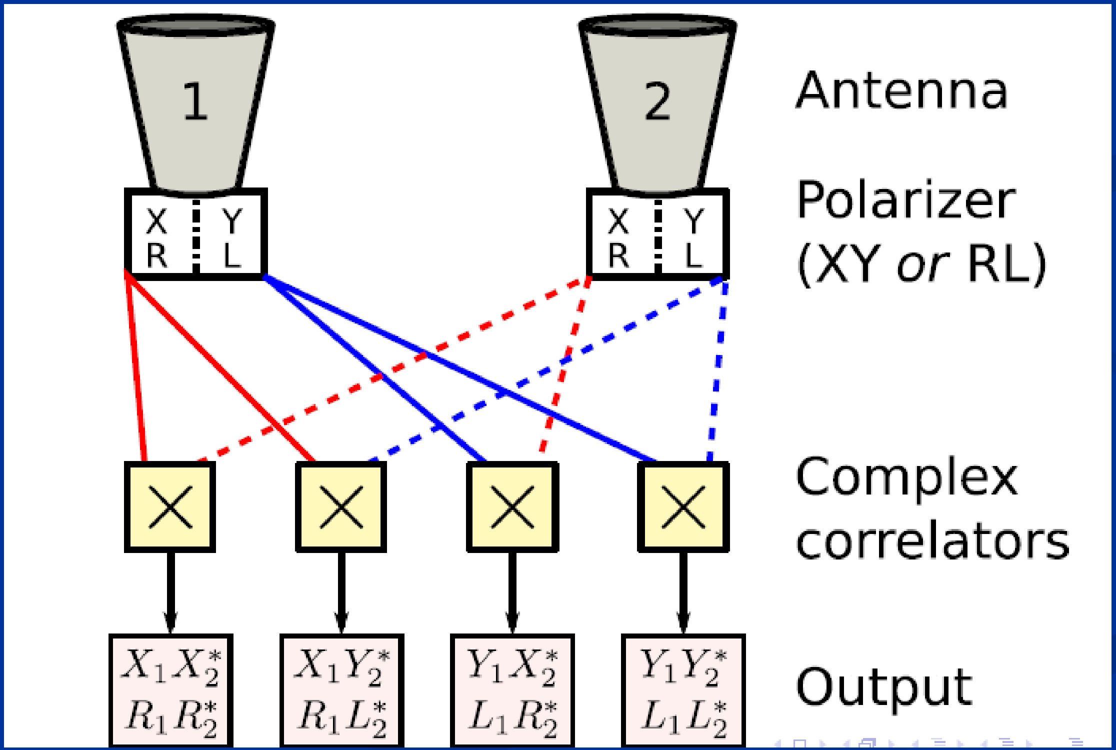
Circular



Linear

# Polarized cross-correlations

- The receiver measures two (nominally) orthogonal polarization states, usually right and left circular or crossed linear.
- Then the polarization is described by a 2 x 2 matrix of correlations between components, which we can correlate and image separately.



# Describing cross-correlations

- Correlator output for antennas i and j is the coherency matrix $\mathbf{V}_{ij}$ of four elements (basis states p and q could be either linear or circular).

$$\mathbf{V}_{ij} = \begin{bmatrix} p_i p_j^* & p_i q_j^* \\ q_i p_j^* & q_i q_j^* \end{bmatrix}$$

- Formalism for describing **linear** corrupting effects which mix polarization states.

$$\mathbf{V'}_{ij} = \mathbf{J}_i \mathbf{V}_{ij} \mathbf{J}_j^\dagger$$ († means transpose and complex conjugate)

- **J's** (2 x 2) are Jones Matrices for antennas i and j. This is the **measurement equation**. Important for two reasons:
  - Invertible to get the true visibilities
  - Jones matrices can be expressed as products of individual Jones matrices for different corrupting effects

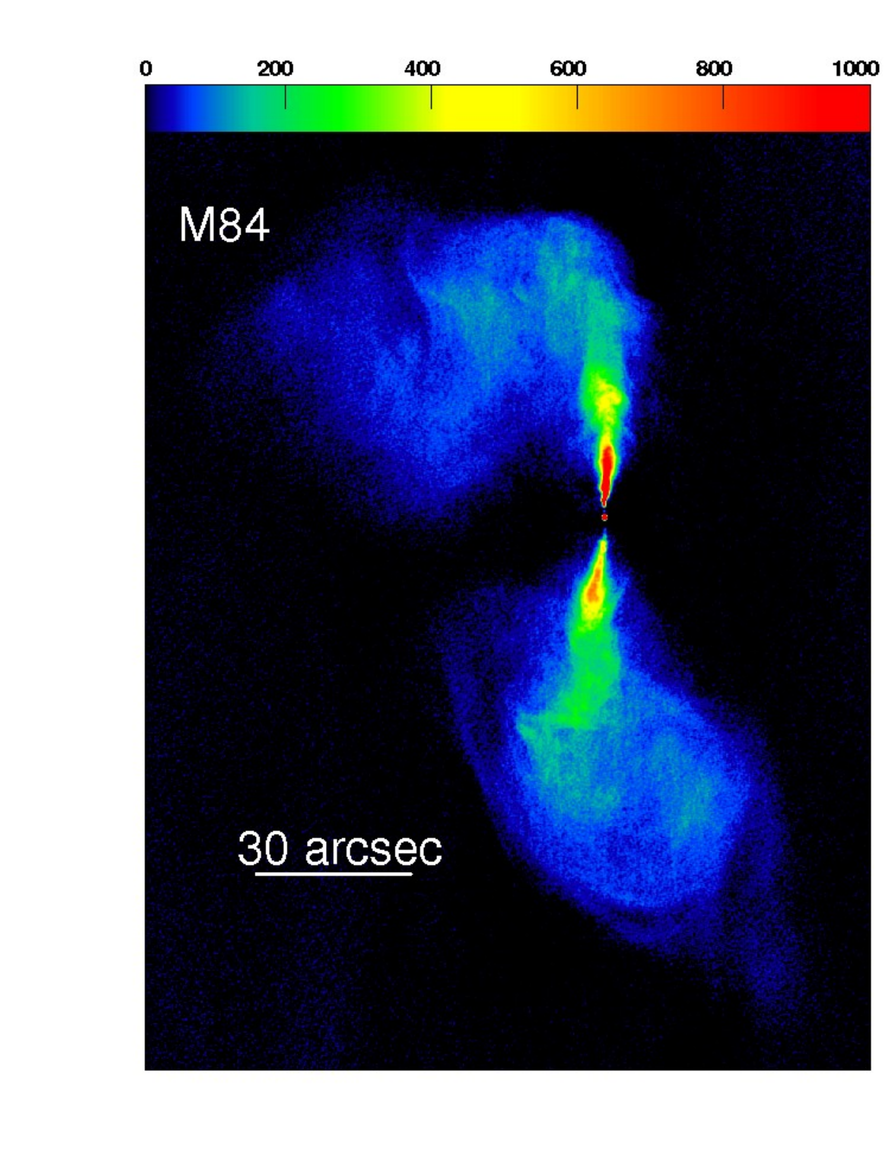
# Dictionary of Jones matrices

- *[S is the sampling function]*
- *F* = ionospheric effects
- *T* = tropospheric effects
- *P* = parallactic angle
- *X* = linear polarization position angle
- *E* = antenna voltage pattern
- *D* = polarization leakage
- *G* = Time-variable *g*ain
- *B* = bandpass response
- *K* = geometric compensation

$$V = MKBG\int DEXPTFS\ I(l,m)\ \exp[-2\pi i(ul+vm)\ dldm$$

Assumes that all corrupting effects are antenna-based

# The End



# References and thanks


- Synthesis Imaging in Radio Astronomy II (ASP Conference Series 180, eds Taylor, Carilli & Perley, 1999), especially lecture 1 by B. Clark.
- On-line lectures of recent NRAO Summer School (2012) www.aoc.nrao.edu/events/synthesis/2012/lectures.shtml
- Born & Wolf, Principles of Optics (1980) [for coherence functions]
- Thomson, Moran & Swenson (2000), Interferometry and Synthesis in Radio Astronomy [for a more hardware-orientated point of view]


Thanks for illustrations to: Rick Perley, Anita Richards