

# Digital Signal Processing and Beamforming

## Tutorial Lecture

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Design of Wideband Receiving Array Systems  
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A.J. Boonstra, ASTRON, boonstra@astron.nl



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# Notation: signals and random variables

- signal:  $s(t)$  or  $s$ , parameter  $t$  often omitted
- random variable:  $x(t)$  or  $x$ , parameter  $t$  often omitted
- probability density function:  $P(x)$

e.g. white Gaussian noise (WGN):  $P(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \mathcal{E}\{x\})^2}{2\sigma^2}}$

- expected value:  $\mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} x(t)P(x)dt$
- correlation:  $\mathbf{R}_{xy} = \mathcal{E}\{x(t)\overline{y(t)}\}$

# Notation: scalars, vectors and matrices

- scalar, lower case or upper case, not bold:  $y$  or  $Y$

- vector, bold lower case:  $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$

- matrix, bold upper case:  $\mathbf{R} = \begin{pmatrix} R_{1,1} & \cdots & R_{1,p} \\ \vdots & \ddots & \vdots \\ R_{p,1} & \cdots & R_{p,p} \end{pmatrix}$

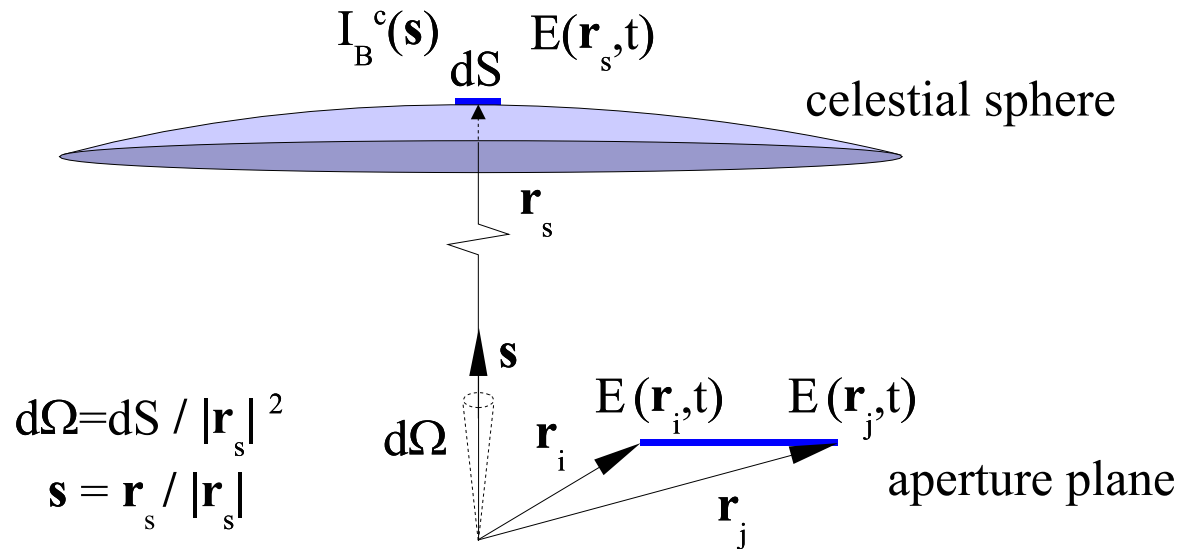
- estimated value:  $\hat{g}$

# Notation: operators

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- complex conjugate:  $\bar{y} = \text{Re}\{y\} - \text{Im}\{y\}$
- transpose:  $\mathbf{x}^t = (x_1, \dots, x_p)$
- Hermitian or complex conjugate transpose:  $\mathbf{R}^H = \overline{\mathbf{R}^t}$
- conversion from vector  $\mathbf{d}$  (or row) to diagonal matrix:  
 $\text{diag}(\mathbf{d}) = \mathbf{D}$

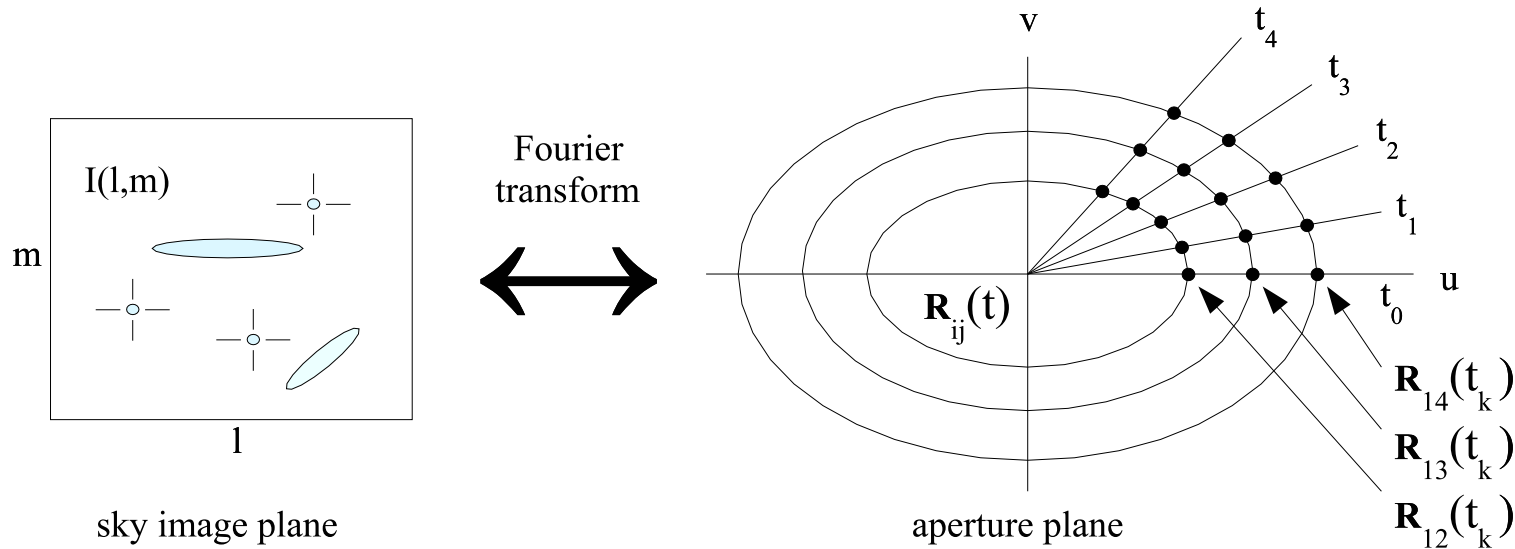
# Spatial coherency and interferometry



Spatial coherence function:  $V(\mathbf{r}_i, \mathbf{r}_j) = \mathcal{E}\{E(\mathbf{r}_i, t)\overline{E(\mathbf{r}_j, t)}\}$ .  
Denote the intensity sky source distribution (continuous) by  $I_B^c$ , then:

$$V(\mathbf{r}_i, \mathbf{r}_j) = \int_{\text{celestial sphere}} I_B^c(\mathbf{s}) e^{-2\pi j \nu c^{-1} \mathbf{s}^t (\mathbf{r}_i - \mathbf{r}_j)} d\Omega$$

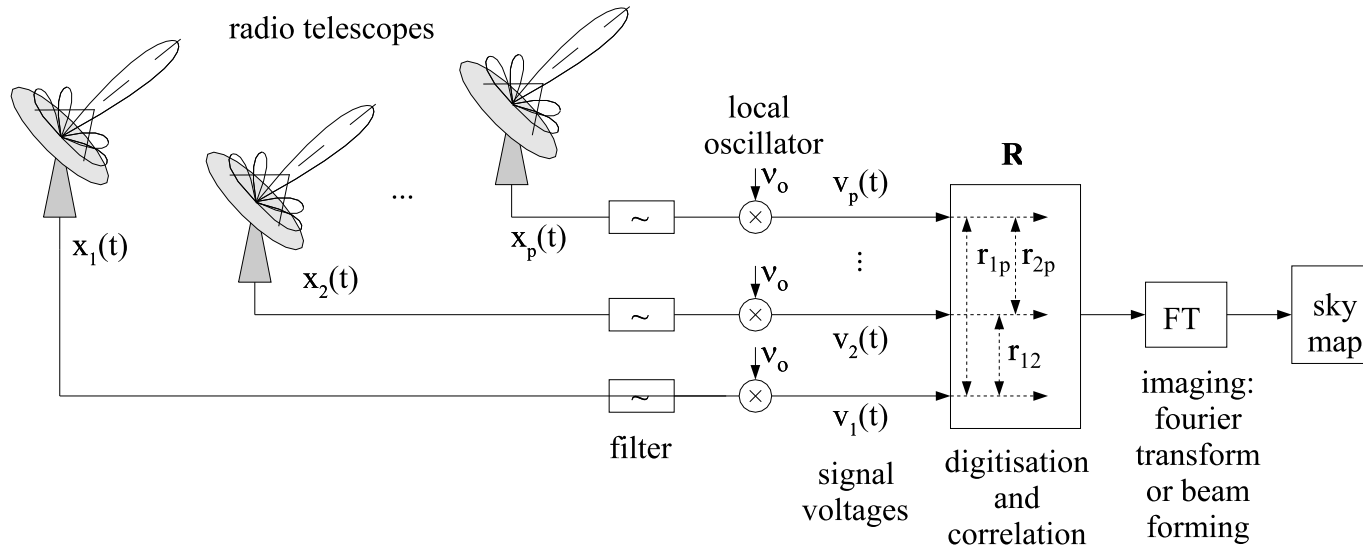
# Imaging



*Van Cittert-Zernike* theorem: the brightness distribution  $I_b(\mathbf{s})$  and the complex visibility function  $V(\mathbf{r}_i, \mathbf{r}_j)$  form a two-dimensional Fourier-transform pair:

$$I_b(\mathbf{s}) \xleftrightarrow{2\text{-D FT}} V(\mathbf{r}_i, \mathbf{r}_j)$$

# Obtaining data

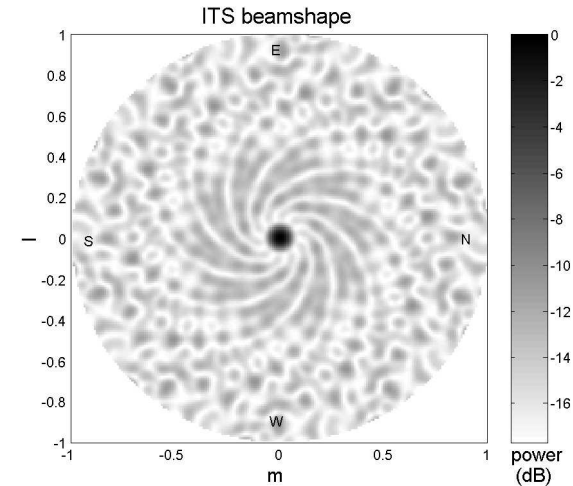
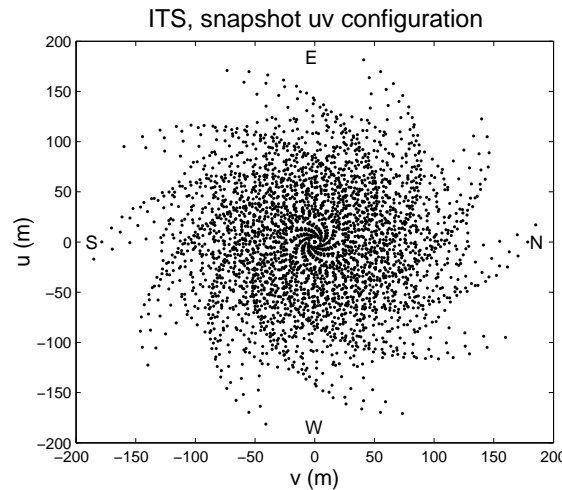
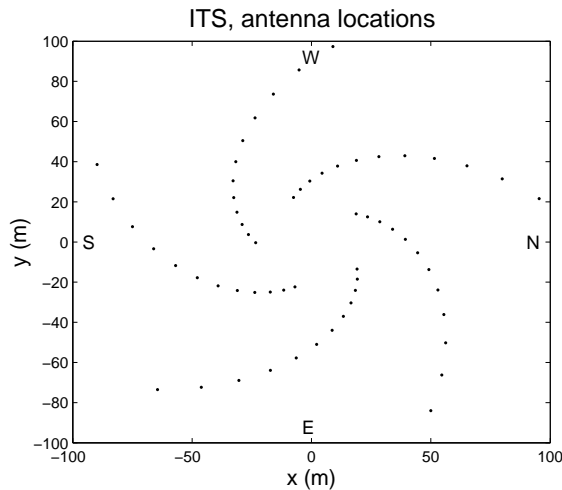


$$\text{Recall: } V(\mathbf{r}_i, \mathbf{r}_j) = \int_{\text{celestial sphere}} I_B^c(\mathbf{s}) e^{-2\pi j \nu c^{-1} \mathbf{s}^t (\mathbf{r}_i - \mathbf{r}_j)} d\Omega$$

The covariance matrix  $\mathbf{R}$  is a measure of the visibility, including (a.o.) antenna gains  $\mathbf{A}(\mathbf{s})$  and electronic gains  $\mathbf{G}$ :

$$R(\mathbf{r}_i, \mathbf{r}_j) = G_i \overline{G_j} \int_{\text{celestial sphere}} \mathcal{A}_i(\mathbf{s}) \overline{\mathcal{A}_j(\mathbf{s})} I_B^c(\mathbf{s}) e^{-2\pi j \nu c^{-1} \mathbf{s}^t \mathbf{b}_{ij}} d\Omega$$

# LOFAR ITS, configuration & beam shape



antenna locations:

$$\mathbf{R} = \begin{pmatrix} x_1 & y_1 & z_1 \\ \dots & \dots & \dots \\ x_p & y_p & z_p \end{pmatrix}$$

antenna signals:

$$\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]^t$$

spatial sample points  
of coherencies (uv-  
points)

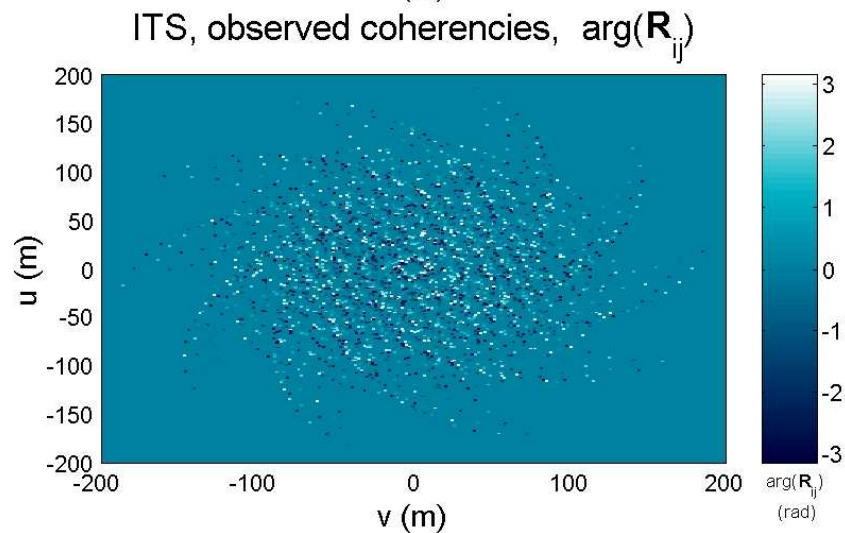
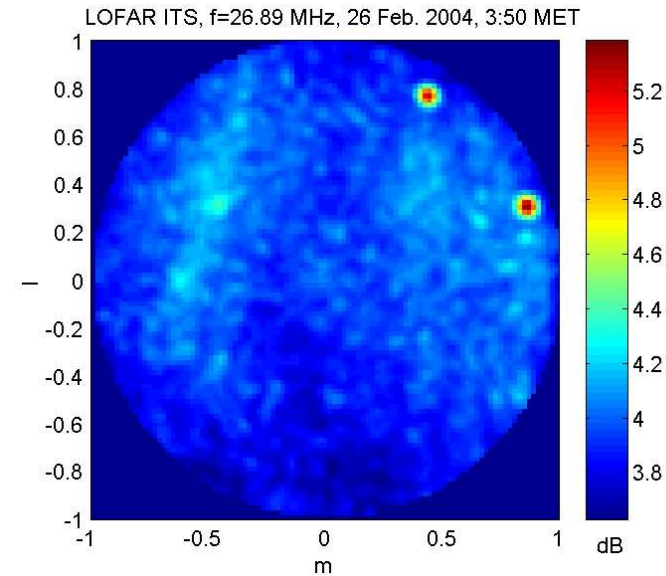
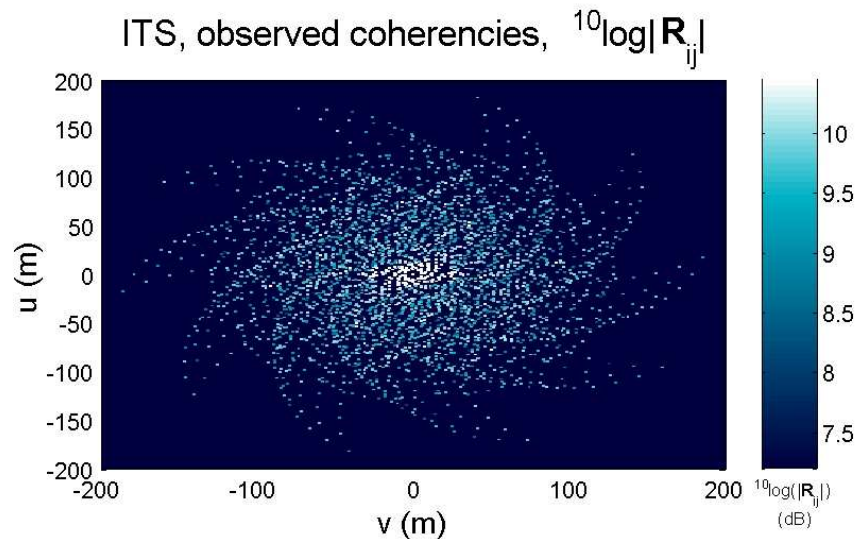
$$\mathbf{R} = \mathcal{E}\{\mathbf{x}\mathbf{x}^H\}$$

$p \times p$  matrix

PSF, or beam shape  
zenith position.  
imaging by beam  
forming or FT

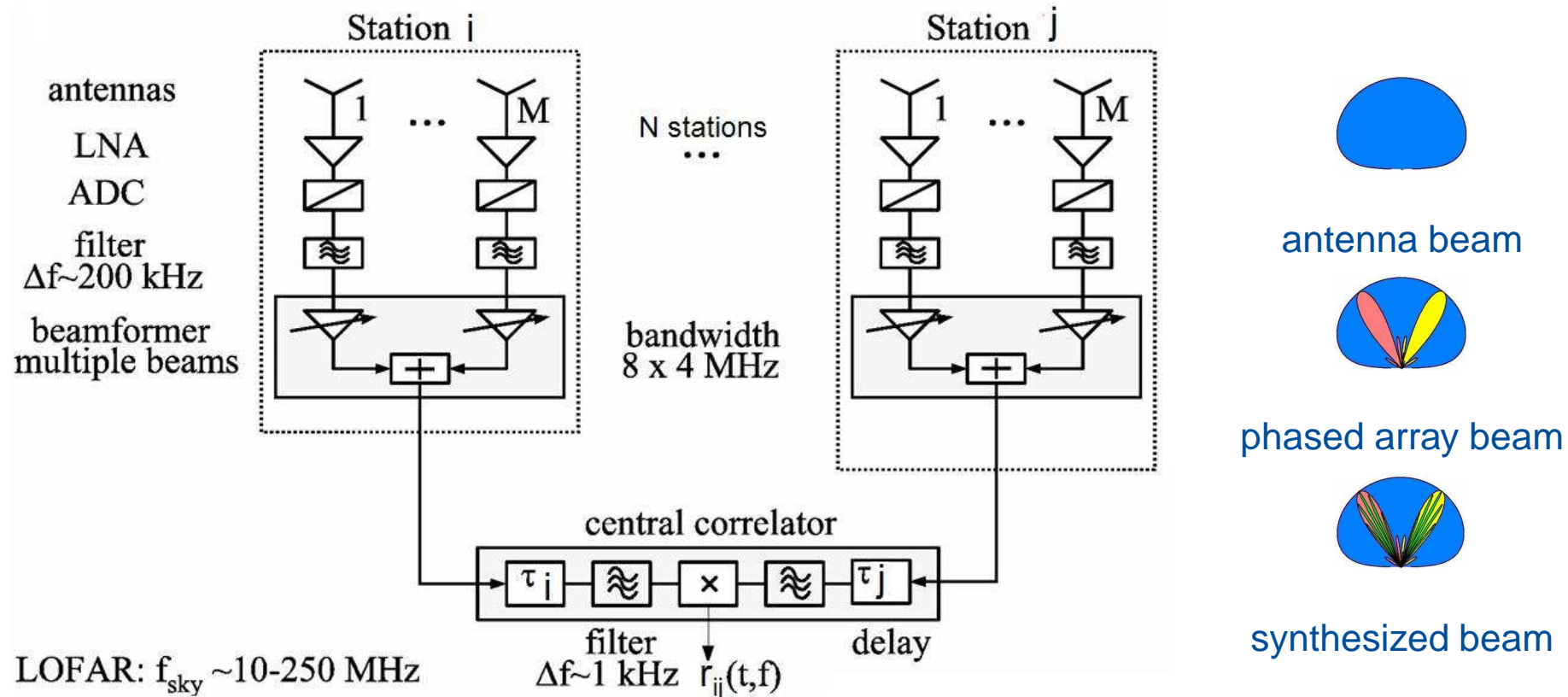
$$P = \mathcal{E}\{\|\mathbf{w}^H \mathbf{x}\|^2\} = \mathbf{w}^H \mathbf{R} \mathbf{w}$$

# LOFAR ITS, coherencies and sky maps



LOFAR ITS test station, observed coherencies (left) and corresponding sky map (upper figure). Sky maps are obtained by inverse FT or by beamforming (as in this case: MVDR beamformer).

# LOFAR, beamforming and correlation



Broadband vs narrowband processing:  
in LOFAR both types, broadband mainly in analog domain.  
Tradeoffs: cost, flexibility, narrowband limits (e.g. beam squint).

# Narrow band assumptions

If for a band-limited signal  $y(t)$ ,

$$\Delta\nu \ll \nu_c$$

then  $y(t) = m(t)e^{2\pi j\nu_c t}$ , where  $m(t)$  is the complex envelope (baseband signal) of the real-valued band-pass signal  $z(t) = \text{Re}\{y(t)\}$ .

Second narrow band condition: propagation delay differences,  $\tau_{ij} = \tau_i - \tau_j$ , along elements  $i$  and  $j$  of the antenna/telescope array are  $\ll$  inverse bandwidth.

$$\Delta\nu \ll (2\pi\tau_{ij})^{-1}$$

Then it can be shown that time delays across array can be approximated by phase shifts of the baseband signal.

# Narrow band assumptions

Recall  $z(t) = \text{Re}\{m(t)e^{2\pi j\nu_c t}\}$ , then

$$z(t - \tau) = \text{Re}\{m(t - \tau)e^{-2\pi j\nu_c \tau} e^{2\pi j\nu_c t}\}$$

which implies  $m_\tau(t) = m(t - \tau)e^{-2\pi j\nu_c \tau}$ .

Define  $m(t) \xleftrightarrow{\mathcal{F}} \mathcal{M}(\nu)$ , and let  $\Delta\nu$  be the bandwidth of  $m(t)$ :

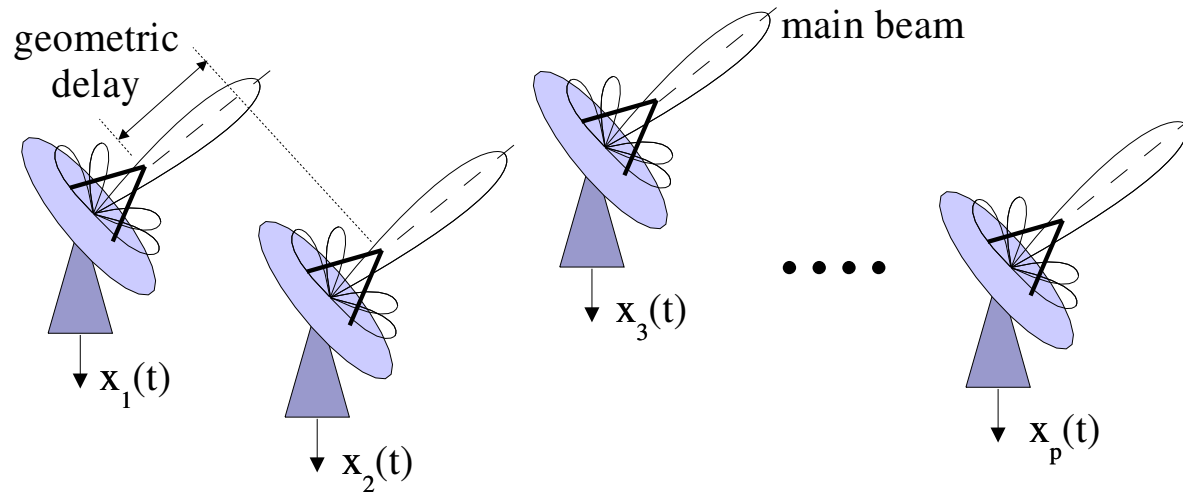
$$m(t) = \int_{-\frac{1}{2}\Delta\nu}^{\frac{1}{2}\Delta\nu} \mathcal{M}(\nu) e^{2\pi j\nu t} d\nu$$

$$m(t - \tau) = \int_{-\frac{1}{2}\Delta\nu}^{\frac{1}{2}\Delta\nu} \mathcal{M}(\nu) e^{-2\pi j\nu \tau} e^{2\pi j\nu t} d\nu$$

If  $\Delta\nu \ll (2\pi\tau_{ij})^{-1}$  then,  $e^{-2\pi j\nu \tau} \approx 1$ , and  $m(t) \approx (t - \tau)$ , hence:

$$m_\tau(t) \approx m(t)e^{-2\pi j\nu_c \tau} \quad \text{QED}$$

# Discrete source model



Consider an array of  $p$  antennas or telescopes with baselines  $\mathbf{b}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ . Signals  $x_i(t)$  are stacked in a vector, the spatially white noise signals  $n_i(t)$  (from LNAs, spill-over etc.) idem:

$$\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]^t$$

$$\mathbf{n}(t) = [n_1(t), \dots, n_p(t)]^t$$

# Discrete source model

Suppose there is one source, either astronomical or a transmitter (RFI) with signal  $s(t)$  and from direction  $\mathbf{s}$

Then

$$\mathbf{x}(t) = \mathbf{a}s(t) + \mathbf{n}(t)$$

and

$$\mathbf{a} = \left[ \gamma_1 e^{-2\pi j \frac{1}{\lambda} \mathbf{r}_1^t \mathbf{s}}, \dots, \gamma_p e^{-2\pi j \frac{1}{\lambda} \mathbf{r}_p^t \mathbf{s}} \right]^t$$

Now define the sample estimate (observational data):

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^H \quad \text{with } \mathbf{x}_n = \mathbf{x}(nT_s)$$

Given i.i.d. noise vector  $\mathbf{n}(t)$  with  $\mathcal{E}\{\mathbf{n}(t)\mathbf{n}(t)^H\} = \sigma_n^2 \mathbf{I}$  ( $\mathbf{I}$  is the identity matrix), and  $\mathcal{E}\{s(t)^2\} = \sigma^2$ :

$$\mathbf{R} = \mathcal{E}\{\hat{\mathbf{R}}\} = \sigma^2 \mathbf{a}\mathbf{a}^H + \sigma_n^2 \mathbf{I}$$

# Discrete source model

Recall for a single source:  $\mathbf{R} = \sigma^2 \mathbf{a} \mathbf{a}^H + \sigma_n^2 \mathbf{I}$ , for multiple sources:

$$\mathbf{R} = \sum_k \sigma_k^2 \mathbf{a}_k \mathbf{a}_k^H + \sigma_n^2 \mathbf{I}$$

Stacking  $q$  sources at directions  $\mathbf{s}_1$  to  $\mathbf{s}_q$  in a matrix  $\mathbf{A}$ :

$$\mathbf{A} = [\mathbf{a}_1(\mathbf{s}_1), \dots, \mathbf{a}_q(\mathbf{s}_q)]$$

and storing the source powers (brightness) in a diagonal matrix  $\mathbf{B}$

$$\mathbf{B} = \text{diag}[\sigma_1^2, \dots, \sigma_q^2]$$

yields for the data model:

$$\mathbf{R} = \mathbf{A} \mathbf{B} \mathbf{A}^H + \sigma_n^2 \mathbf{I}$$

# Discrete source model

Recall the data model:

$$\mathbf{R} = \mathbf{A}\mathbf{B}\mathbf{A}^H + \sigma_n^2 \mathbf{I}$$

Assume:

- there are astronomical sources "s" and interferers "r",
- the noise powers are unequal:  $\mathcal{E}\{\mathbf{n}(t)\mathbf{n}(t)^H\} = \mathbf{D}$  (diagonal),
- define a diagonal electronic gain matrix  $\mathbf{G}$ ,

then the discrete data model becomes:

$$\mathbf{R} = \mathbf{G}(\mathbf{A}_r \mathbf{B}_r \mathbf{A}_r^H + \mathbf{A}_s \mathbf{B}_s \mathbf{A}_s^H) \mathbf{G}^H + \mathbf{D}$$

Note: "whitening" can be done by pre and post multiplying  $\mathbf{R}$  with  $\mathbf{D}^{-\frac{1}{2}}$  or  $\mathbf{R}^{-\frac{1}{2}}$  (yields correlation coefficients).

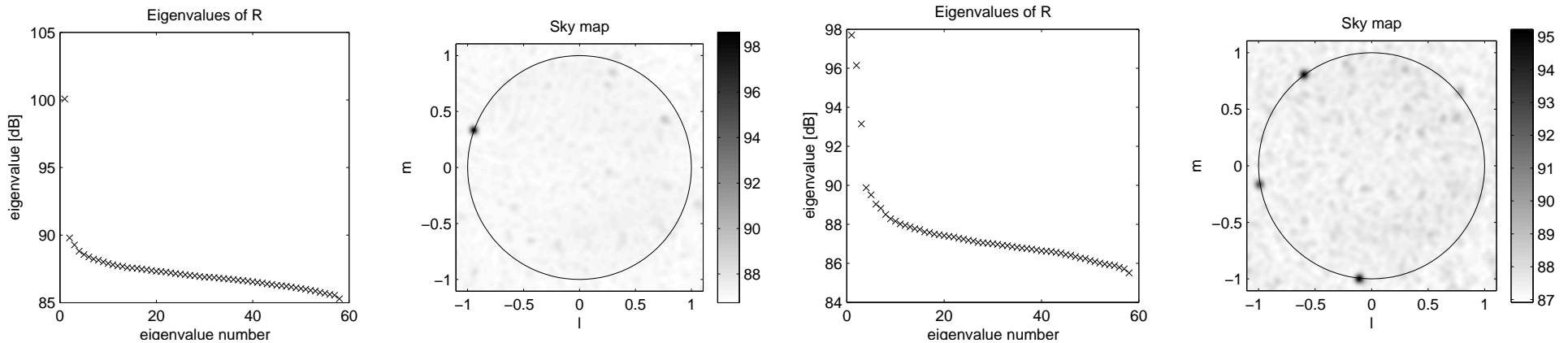
# Subspace analysis

Given  $q$  sources, and an array of  $p$  antennas, then  $\mathbf{R}$  can be decomposed by means of an eigenvalue decomposition such that  $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$ , where  $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_p]^t$  contains the eigenvectors  $\mathbf{u}_i$  ( $\mathbf{U}^H\mathbf{U} = \mathbf{I}$ ), and  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_p)$  contains the eigenvalues. Also:

$$\begin{aligned}\mathbf{R} &= \mathbf{A}_r \mathbf{B}_r \mathbf{A}_r + \sigma_n^2 \mathbf{I}_p \\ &= \mathbf{U}_r \mathbf{\Lambda}_r \mathbf{U}_r^H + \sigma_n^2 (\mathbf{U}_r \mathbf{U}_r^H + \mathbf{U}_n \mathbf{U}_n^H) \\ &= [\mathbf{U}_r \mathbf{U}_n] \begin{bmatrix} \mathbf{\Lambda}_r + \sigma_n^2 \mathbf{I}_q & 0 \\ 0 & \sigma_n^2 \mathbf{I}_{p-q} \end{bmatrix} \begin{bmatrix} \mathbf{U}_r^H \\ \mathbf{U}_n^H \end{bmatrix} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H\end{aligned}$$

The sources are located in the "source subspace"  $\mathbf{U}_r$ , separated from the "noise subspace"  $\mathbf{U}_n$ . Note that all eigenvectors are perpendicular and  $\mathbf{U}_r^H \mathbf{U}_n = \mathbf{0}$ .

# Relation subspace structure and RFI



Relation between multiple transmitters and eigenvalue distribution of covariance matrix  $\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^H$ . Data (not normalized/whitened) are obtained from LOFAR ITS test station. One horizon transmitter (RFI) (left), and three transmitters (right)<sup>(1)</sup>.

Note: for astronomical sources similar relation holds.

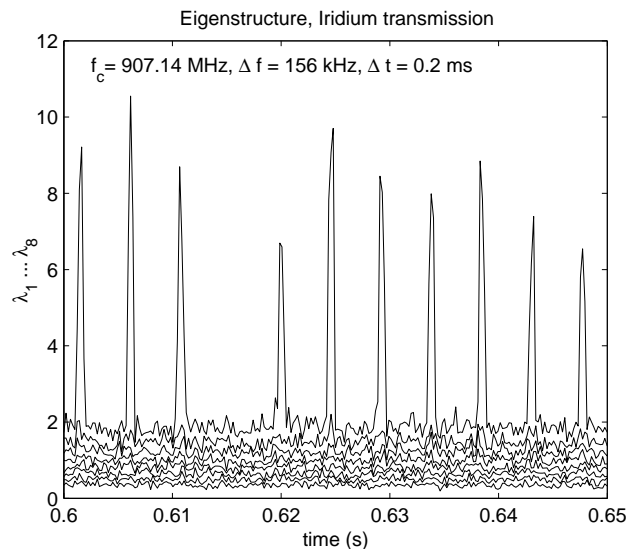
(1) A.J. Boonstra and S. van der Tol, 2004.

# Detection based on subspace structure

For identical noise powers  $\sigma_n^2$ , or for a whitened  $\mathbf{R}$  matrix <sup>(1)</sup>

$$\lambda_{max,min} \approx \sigma^2 \left( 1 \pm \sqrt{\frac{p}{N}} \right)^2$$

This can be used e.g. as detection threshold for intermittent RFI, see figure below showing RFI at the WSRT (whitened normalized dataset).

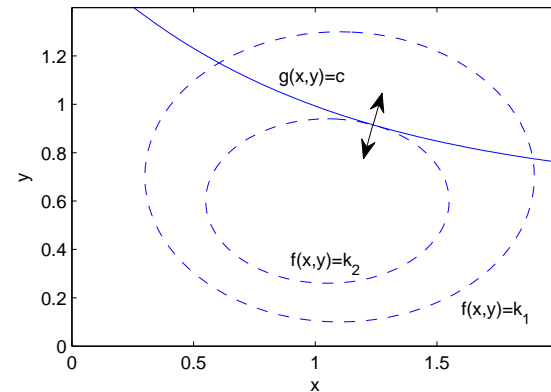


An alternative, computationally less expensive, is to use  $\|\mathbf{R}\|_F^2$  (sum of absolute squares of the elements) <sup>(2)</sup>

<sup>(1)</sup> Edelman 1988. <sup>(2)</sup> Leshem and van der Veen, 2001.

# Lagrange multipliers

Lagrange multipliers: finding extreme of a function  $f(x, y)$  subject to the constraint  $g(x, y) - c = 0$ .



At extremum: gradient  $\nabla_{\perp}$  of  $f$  and  $g$  aligned (arrows in the figure), and tangential component  $\nabla_{\parallel}$  of derivative is zero:  $\nabla f(x, y) = \lambda \nabla(g(x, y) - c)$  where  $\lambda$  is a scaling constant, the Lagrange multiplier. Define  $\mathcal{L} = f(x, y) - \lambda(g(x, y) - c)$ . Then:

$$\frac{\partial}{\partial x} \mathcal{L} = 0, \quad \frac{\partial}{\partial y} \mathcal{L} = 0, \quad \frac{\partial}{\partial \lambda} \mathcal{L} = 0$$

The latter equation is true as it reduces to the original constraint.

# Lagrange multipliers

Generalization to  $N$ -dimensional space.

In  $N$ -dimensional space, with  $\mathbf{x} = [x_1, \dots, x_N]^t$ , and with  $K$  constraints we define the Lagrangian  $\mathcal{L}$ :

$$\mathcal{L}(\mathbf{x}, \lambda_1, \dots, \lambda_K) = f(\mathbf{x}) + \sum_{k=1}^K \lambda_k g(\mathbf{x})$$

where the previous constant  $c$  of  $g(\mathbf{x})$  is now included in the definition of  $g(\mathbf{x})$ . This yields  $N + K$  equations and  $N + K$  unknowns:

$$\nabla_{\mathbf{x}} \mathcal{L} = 0, \quad \nabla_{\boldsymbol{\lambda}} \mathcal{L} = 0$$

with  $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_K]^t$

# MVDR beamformer

Given  $p$  zero-mean antenna element random signals  $\mathbf{x}(t) = [x_1(t) \cdots x_p(t)]^t$ , with  $\mathbf{R} = \mathcal{E}\{\mathbf{x}(t)\mathbf{x}(t)^H\}$ . Let

$$y(t) = \mathbf{w}^H \mathbf{x}(t)$$

be the beamformer output signal with beamformer weights  $\mathbf{w}$ .  
Minimize beamformer signal variance

$$P = \min_{\mathbf{w}} \mathcal{E}\{|y|^2\} = \min_{\mathbf{w}} (\mathbf{w}^H \mathbf{R} \mathbf{w})$$

with constraint of unit gain in look-direction  $\mathbf{s}$ ,  $\mathbf{w}^H \mathbf{a} = 1$  where

$$\mathbf{a} = e^{-2\pi j \mathcal{R} \mathbf{s} f c^{-1}}$$

Now define the Lagrangian:  $\mathcal{L} = \mathbf{w}^H \mathbf{R} \mathbf{w} + \lambda(\mathbf{w}^H \mathbf{a} - 1)$

# MVDR beamformer

Recall  $\mathcal{L} = \mathbf{w}^H \mathbf{R} \mathbf{w} + \lambda (\mathbf{w}^H \mathbf{a} - 1)$

Using  $\nabla_{\mathbf{w}} (\mathbf{w}^H \mathbf{R} \mathbf{w}) = \mathbf{R}^t \bar{\mathbf{w}}$ , and  $\nabla_{\mathbf{w}} (\mathbf{a}^H \mathbf{w}) = \bar{\mathbf{a}}$  <sup>(1)</sup> yields:

$$\nabla_{\mathbf{w}} \mathcal{L} = \mathbf{R}^t \bar{\mathbf{w}} + \lambda \bar{\mathbf{a}} = \mathbf{0}$$

This results in  $\mathbf{w} = -\bar{\lambda} \mathbf{R}^{-1} \mathbf{a}$ ; inserting it in constraint equation leads to

$$\lambda = -\frac{1}{\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a}}$$

and to the MVDR/Capon beamformer weights:

$$\mathbf{w} = \frac{\mathbf{R}^{-1} \mathbf{a}}{\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a}}$$

A treatise on real vs complex derivative definitions and relations can be found in <sup>(1)</sup>.

<sup>(1)</sup> cf. Kay 1998, Moon 2000

# Conventional beamformer

Given  $p$  zero-mean antenna element random signals  $\mathbf{x}(t) = [x_1(t) \cdots x_p(t)]^t$ , with  $\mathbf{R} = \mathcal{E}\{\mathbf{x}(t)\mathbf{x}(t)^H\}$ . Let

$$y(t) = \mathbf{w}^H \mathbf{x}(t)$$

be the beamformer output signal with beamformer weights  $\mathbf{w}$ . Maximize beamformer output power

$$P = \max_{\mathbf{w}} \mathcal{E}\{|y|^2\} = \max_{\mathbf{w}} (\mathbf{w}^H \mathbf{R} \mathbf{w})$$

with constraint of unit norm of  $\mathbf{w}$ . Solving the gradient equations yields:

$$\mathbf{w} = \frac{\mathbf{a}}{\sqrt{\mathbf{a}^H \mathbf{a}}}$$

The vector  $\mathbf{w}$  can be interpreted as a spatial filter matched to the impinging signal.

# Beamformer spatial spectra

Assuming observational data is used, the spatial spectrum is given by inserting the weight into the beamformer output power formula:

$$P_{conv}(\mathbf{s}) = \frac{\mathbf{a}^H \hat{\mathbf{R}} \mathbf{a}}{\mathbf{a}^H \mathbf{a}}$$

and

$$P_{mvdr}(\mathbf{s}) = \frac{1}{\mathbf{a}^H \hat{\mathbf{R}}^{-1} \mathbf{a}}$$

where the steering vector  $\mathbf{a}$ , as before, is defined by

$$\mathbf{a} = e^{-2\pi j \mathcal{R} \mathbf{s} \lambda^{-1}}$$

with  $\mathcal{R} = [\mathbf{b}_1, \dots, \mathbf{b}_p]^t$ .

# Music beamformer

Recall for the MVDR beamformer:  $P_{mvdr}(\mathbf{s}) = \frac{1}{\mathbf{a}^H \hat{\mathbf{R}}^{-1} \mathbf{a}}$

Recall also the data model:

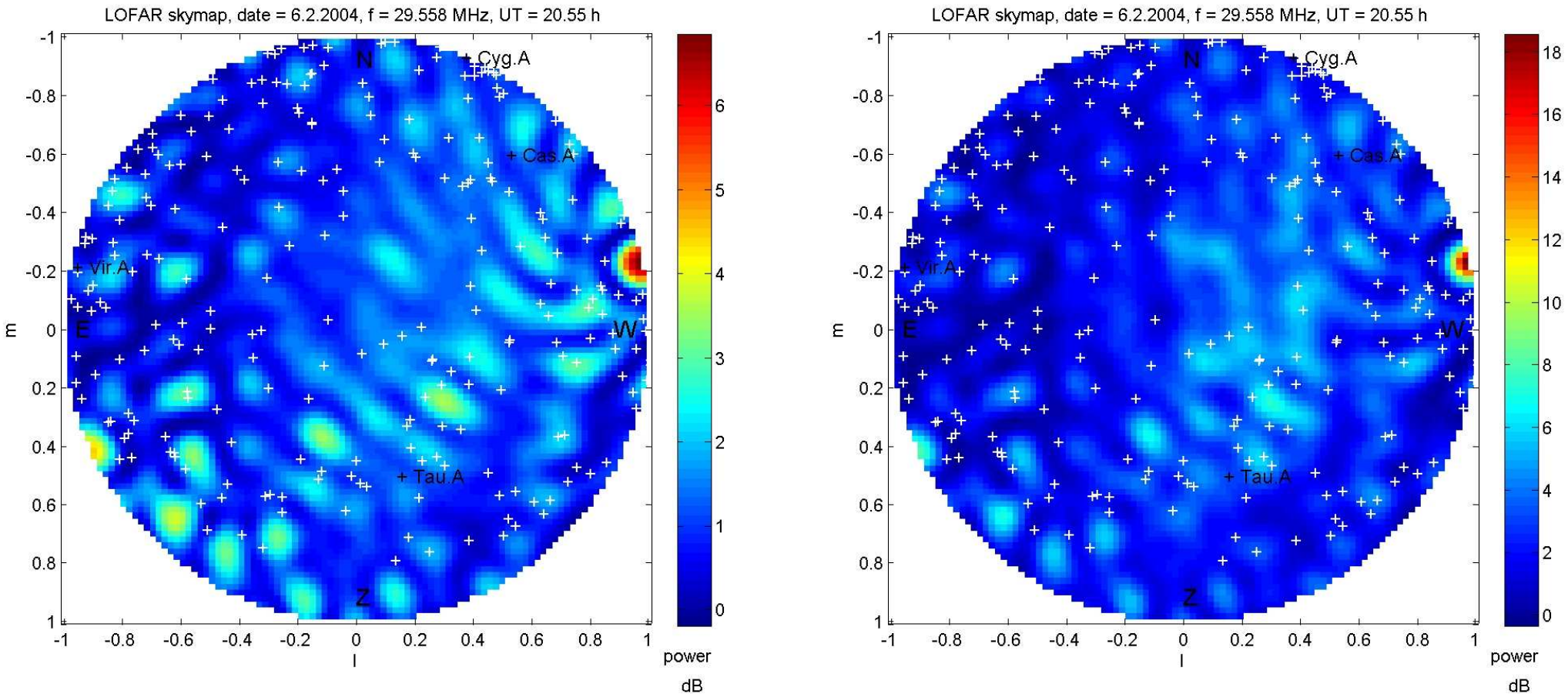
$$\mathbf{R} = \mathbf{A} \mathbf{B} \mathbf{A}^H + \sigma_n^2 \mathbf{I} = \mathbf{U}_r \mathbf{\Lambda}_r \mathbf{U}_r^H + \sigma_n^2 \mathbf{U}_n \mathbf{U}_n^H$$

As  $\mathbf{U}_n^H \mathbf{a} = 0$  for all  $\mathbf{a}$  in  $\mathbf{A}$ , define the ad-hoc "MUSIC" beamformer spatial spectrum by:

$$P_{music}(\mathbf{s}) = \frac{\mathbf{a}^H \mathbf{a}}{\mathbf{a}^H \mathbf{\Pi}^\perp \mathbf{a}}$$

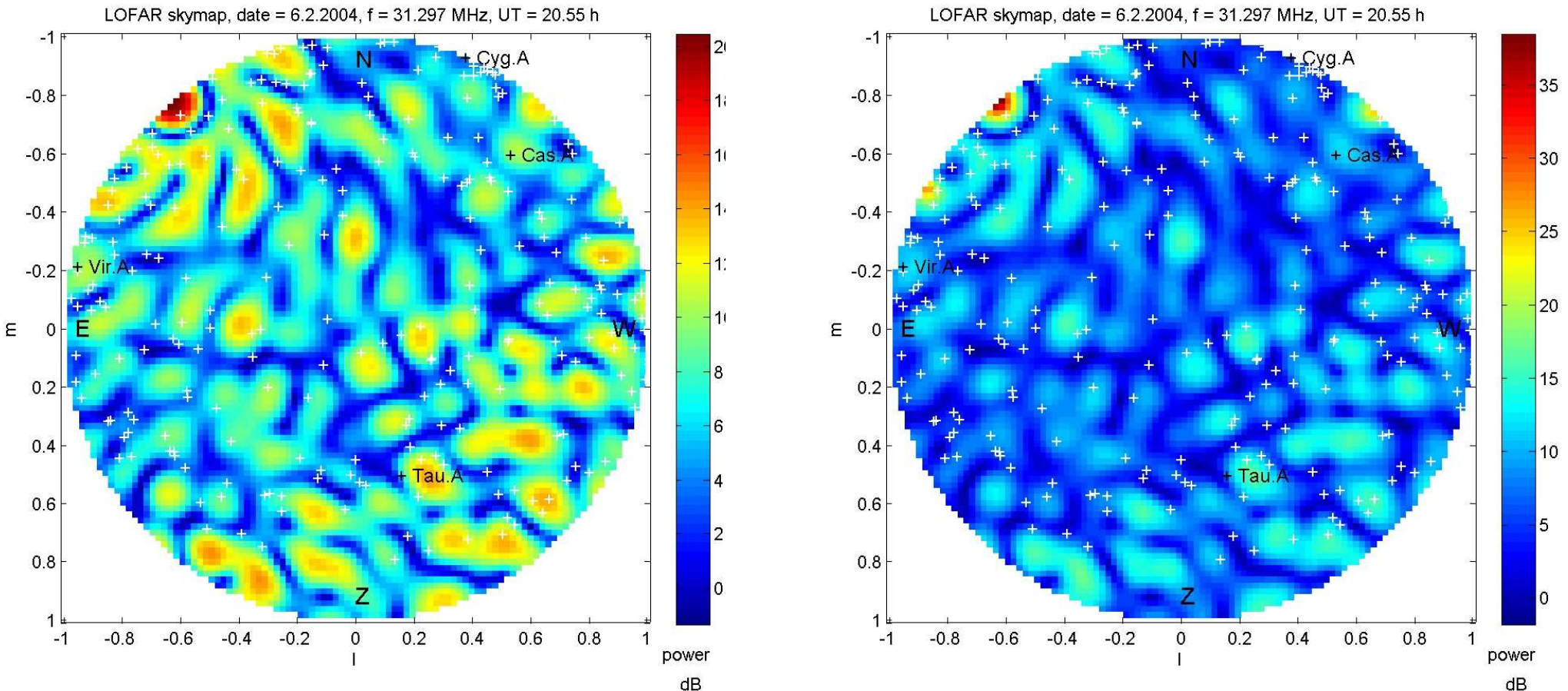
with  $\mathbf{\Pi}^\perp = \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H$ , with  $\hat{\mathbf{U}}_n$  the noise subspace. This leads (as with the MVDR) to an improved sidelobe suppression and smaller beamwidth as compared to the conventional beamformer. Drawback: these are data dependent and computation scales with  $p^3$ , as compared to  $p$  for the conventional beamformer.

# Conventional vs subspace beamforming



Conventional beamforming (left) vs MUSIC beamforming (right), using LOFAR ITS data. The sky map is dominated by a transmitter at the horizon, note the difference in sidelobe level and the difference in beamwidth. The plusses indicate the location of astronomical 3C sources.

# Conventional vs subspace beamforming



Conventional beamforming (left) vs MUSIC beamforming (right), using LOFAR ITS data. The sky map is dominated by a transmitter at the horizon, note the difference in sidelobe level and the difference in beamwidth. The plusses indicate the location of astronomical 3C sources.

# Spatial projection filtering

Define the data model, using a separate covariance matrix for the astronomical visibilities  $\mathbf{R}_v$ , and assuming one RFI source:

$$\mathbf{R} = \mathbf{R}_v + \sigma_n^2 \mathbf{I} + \sigma^2 \mathbf{a} \mathbf{a}^H$$

Projection filtering, define the projection matrix  $\mathbf{P}$ :

$$\mathbf{P} = \mathbf{I} - \mathbf{a}(\mathbf{a}^H \mathbf{a})^{-1} \mathbf{a}^H$$

then  $\mathbf{P} \mathbf{a} = \mathbf{0}$ . Define the filtered matrix  $\tilde{\mathbf{R}}$  by

$$\tilde{\mathbf{R}} = \mathbf{P} \hat{\mathbf{R}} \mathbf{P}, \quad \text{then} \quad \mathcal{E}\{\tilde{\mathbf{R}}\} = \mathbf{P}(\mathbf{R}_v + \sigma_n^2 \mathbf{I}) \mathbf{P}$$

The remaining distortions can be corrected, under certain stationarity conditions<sup>(1)</sup>.

(1) Boonstra, 2005.

# Spatial projection filtering

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Recall the spatial projection filter

$$\mathbf{P} = \mathbf{I} - \mathbf{a}(\mathbf{a}^H \mathbf{a})^{-1} \mathbf{a}^H$$

This can easily be extended to multiple ( $q$ ) interferes:

$$\mathbf{P} = \mathbf{I} - \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$$

where, as before,

$$\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_q]$$

The filter can be applied to covariance (correlation) data, but also is a beamformer by multiplying it with the beamformer weights.

# Spatial subtraction filtering

Recall the data model:

$$\mathbf{R} = \mathbf{R}_v + \sigma_n^2 \mathbf{I} + \sigma^2 \mathbf{a} \mathbf{a}^H$$

The subtraction filtered covariance matrix  $\tilde{\mathbf{R}}$  is given by:

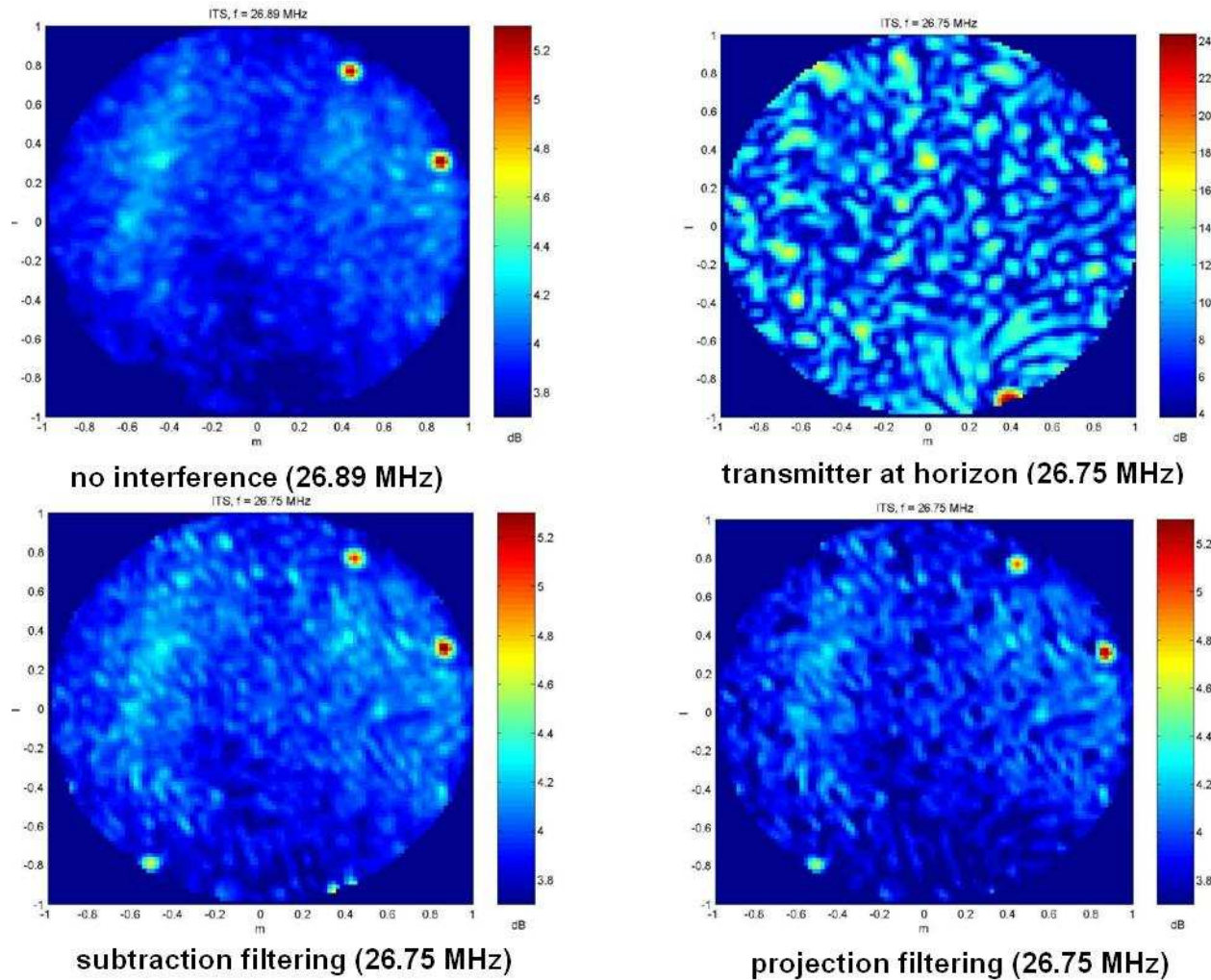
$$\tilde{\mathbf{R}} = \hat{\mathbf{R}} - \hat{\sigma}^2 \hat{\mathbf{a}} \hat{\mathbf{a}}^H$$

where  $\hat{\sigma}^2$  and  $\hat{\mathbf{a}}$  need to be estimated. Especially estimating  $\hat{\sigma}^2$  is computationally expensive (needs short feedback loops).

Theoretical estimates of effectiveness of spatial filters have been found<sup>1</sup>.

(1) Boonstra, 2005.

# Spatial filtering results



LOFAR ITS spatial filtering results.

# References

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## Textbooks on array processing, detection, and estimation techniques

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