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KLT of radio signals from relativistic spaceships in uniform and decelerated motion

18.1 INTRODUCTION

It is well known that in special relativity two time variables exist: the coordinate time t, which is the time measured in the fixed reference frame, and the proper time τ, which is the time shown by a clock rigidly connected to the moving body. They are related by

$$\tau(t) = \int_0^t \sqrt{1 - \frac{v^2(s)}{c^2}}\, ds \tag{18.1}$$

where $v(t)$ is the body velocity and c is the speed of light (see [1, p. 44]).

The remainder of this book, starting with the present chapter, is devoted to the relativistic interpretation of Brownian motion whose time variable is the proper time, $B(\tau)$, rather than the coordinate time, $B(t)$ and to find the KLT of $B(\tau)$. The bulk of these results was given by the author in a purely mathematical form, with no reference to relativity, in [2]. The KLT is also explained in detail in Chapter 16 and Chapters 21–25. However, to enable the reader to read Chapters 18–14 independently of Chapter 16 and Chapters 21–25, a summary of that work is now given in a form suitable for the physical developments that will follow in Chapters 12–14.

Consider standard Brownian motion (Wiener–Lévy process) $B(t)$, with mean zero, variance t, and initial condition $B(0) = 0$, as described in Chapter 21.

A white noise integral is the process $X(t)$ defined by

$$X(t) = \int_0^t f(s)\, dB(s) \tag{18.2}$$

C. Maccone, *Mathematical SETI*.

where $f(t)$ is assumed to be continuous and non-negative. Evidently, $X(0) = 0$, and it can be proved (see (21.35) or, equivalently, [3, pp. 84–87]) that

$$X(t) = B\left(\int_0^t f^2(s)\,ds\right). \tag{18.3}$$

Thus, $X(t)$ is a time-rescaled Gaussian process, with mean zero and

$$E\{X(t_1)X(t_2)\} = \int_0^{t_1 \wedge t_2} f^2(s)\,ds \tag{18.4}$$

as autocorrelation (covariance); $t_1 \wedge t_2$ denotes the minimum (smallest) t_1 and t_2.

Now the KLT theorem (see [4, pp. 262–271]) states that

$$X(t) = \sum_{n=1}^{\infty} Z_n \phi_n(t) \quad (0 \le t \le T) \tag{18.5}$$

where (1) the functions $\phi_n(t)$ are the autocorrelation eigenfunctions to be found from

$$\int_0^T E\{X(t_1)X(t_2)\}\phi_n(t_2)\,dt_2 = \lambda_n \phi_n(t_1) \tag{18.6}$$

where the constants λ_n are the corresponding eigenvalues; and (2) the Z_n are orthogonal random variables, with mean zero and variance λ_n; that is:

$$E\{Z_m Z_n\} = \lambda_n \delta_{mn}. \tag{18.7}$$

This theorem is valid for any continuous-parameter second-order process with mean zero and known autocorrelation. The series (18.5) converges in mean square, and uniformly in t. Finally, if $X(t)$ is Gaussian—as in Equations (18.2) and (18.3)—the random variables Z_n are also Gaussian, and since they are orthogonal they are independent.

After these preliminaries, we can state the main result of [2] (Maccone First KLT Theorem, fully proven in Chapter 22).

The white noise integral (18.2), or the equivalent time-rescaled Gaussian process (18.3), has the KLT expansion:

$$X(t) = \sum_{n=1}^{\infty} Z_n N_n \sqrt{f(t) \int_0^t f(s)\,ds} \cdot J_{\nu(t)}\left(\gamma_n \frac{\displaystyle\int_0^t f(s)\,ds}{\displaystyle\int_0^T f(s)\,ds}\right). \tag{18.8}$$

Here

(1) the order of the Bessel functions $\nu(t)$ is not a constant, but the time function

$$\nu(t) = \sqrt{-\frac{\chi^3(t)}{f^2(t)} \cdot \frac{d}{dt}\left[\frac{\chi'(t)}{f^2(t)}\right]} \tag{18.9}$$

with

$$\chi(t) = \sqrt{f(t) \int_0^t f(s)\, ds}. \tag{18.10}$$

(2) The constants γ_n are the (increasing) positive zeros of

$$\chi'(T) \cdot J_{\nu(T)}(\gamma_n) + \chi(T) \cdot \left[\frac{f(T) \cdot \gamma_n}{\int_0^T f(s)\, ds} J'_{\nu(T)}(\gamma_n) + \frac{\partial J_{\nu(T)}(\gamma_n)}{\partial \nu} \nu'(T) \right] = 0. \tag{18.11}$$

In general, (18.11) can only be solved numerically.

(3) The normalization constants N_n follow from the normalization condition

$$N_n^2 \left[\int_0^T f(s)\, ds \right]^2 \cdot \int_0^1 x[J_{\nu((x))}(\gamma_n x)]^2\, dx = 1 \tag{18.12}$$

where the new Bessel functions order $\nu((x))$ is (18.9) changed by aid of the transformation

$$\int_0^t f(s)\, ds = x \int_0^T f(s)\, ds.$$

(4) The eigenvalues are determined by

$$\lambda_n = \left[\int_0^T f(s)\, ds \right]^2 \frac{1}{(\gamma_n)^2}. \tag{18.13}$$

(5) The Gaussian random variables Z_n are independent and orthogonal, and have zero mean and variance λ_n.

The proof of this theorem may be sketched as follows: first, the Volterra-type integral equation (18.6) is transformed into a differential equation with two boundary conditions; and, second, the latter is reduced to the standard Bessel differential equation by means of two changes of variables. The full proof is given in Chapter 22.

Let us now go back to relativity. Since from (18.3) it plainly appears that the rescaled time of the new Brownian motion is given by

$$\int_0^t f^2(s)\, ds \tag{18.14}$$

we merely have to equate (18.1) and (18.14) to get the relationship among the arbitrary time-rescaling function $f(t)$ and the arbitrary body velocity $v(t)$:

$$\int_0^t f^2(s)\, ds = \int_0^t \sqrt{1 - \frac{v^2(s)}{c^2}}\, ds. \tag{18.15}$$

By differentiating and taking the positive square root, it follows that:

$$f(t) = \left[1 - \frac{v^2(t)}{c^2}\right]^{\frac{1}{4}}. \tag{18.16}$$

This formula is the starting point to study the KLT expansion (18.8) for a relativistic body, like a relativistic spacecraft or spaceship moving in a radial direction away or towards the Earth.

Inversion of (18.16) leads at once to:

$$v(t) = c\sqrt{1 - f^4(t)}. \tag{18.17}$$

Now, the reality of the motion requires the radicand to be non-negative, whence, taking the positive sign in front of all square roots, we find

$$f(t) \leq 1. \tag{18.18}$$

This is the fundamental upper bound imposed on the "arbitrary" function $f(t)$ by special relativity. In other words, as the speed of light can in no case be exceeded, so $f(t)$ must not exceed 1.

As already pointed out, the lower bound on $f(t)$, required by the presence of the radicals in (18.8) and (18.10), is zero. Therefore

$$0 \leq f(t) \leq 1 \quad (0 \leq t \leq T) \tag{18.19}$$

is the physical range of the (otherwise arbitrary) function $f(t)$.

We also need to point out the Newtonian limit of the results. By this we mean the limit as $c \to \infty$. Then, as we see from (18.16),

$$\lim_{c \to \infty} f(t) = 1 \tag{18.20}$$

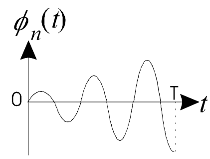
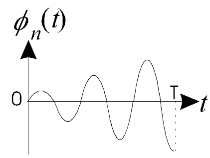
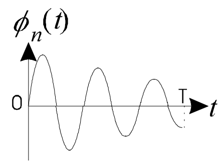
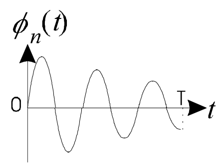
and the time-rescaled process under consideration reduces to standard Brownian motion, $B(t)$. This agrees, of course, with (18.1), stating that the proper time τ becomes the same as the coordinate time t in the Newtonian limit $c \to \infty$.

Finally, we want to hint at how the shape of the eigenfunctions $\phi_n(t)$ may be determined even without knowing their analytical expression. This possibility is a consequence of the Sonine–Pólya theorem, which is explored in Section 22.5, for the non-relativistic case. The reader is referred there for the details, and here we merely confine ourselves to the relativistic version of the results. From (18.16) and (22.61) one finds:

$$\begin{aligned} \frac{d \ln f(t)}{dt} &= \frac{1}{4}\frac{d}{dt}\left[\ln\left(1 - \frac{v^2(t)}{c^2}\right)\right] = -\frac{1}{2c^2} \cdot \frac{v(t)\dfrac{dv(t)}{dt}}{1 - \dfrac{v^2(t)}{c^2}} \\ &= (\text{negative}) \cdot v(t)\frac{dv(t)}{dt}. \end{aligned} \tag{18.21}$$

Thus, not only the velocity $v(t)$, but also its derivative (i.e., acceleration taken with respect to the coordinate time, t) determines the shape (i.e., the stability) of the $\phi_n(t)$. The resulting Table 18.1 follows from this and Table 22.1.

Table 18.1. Stability criterion for the relativistic eigenfunctions $\phi_n(t)$.

Sign of the velocity $v(t)$	Sign of the coordinate acceleration $dv(t)/dt$	Shape of the KL eigenfunctions $\phi_n(t)$	Description when T is finite	Description when T is infinite
Positive	Negative		Divergent	Asymptotic unstable
Negative	Positive		Divergent	Asymptotic unstable
Positive	Positive		Convergent	Asymptotic stable
Negative	Negative		Convergent	Asymptotic stable

18.2 UNIFORM MOTION

The simplest possible case of (18.16) is when the velocity $v(t)$ is a constant (i.e., the body's motion is uniform). Then $f(t)$ is a constant K as well

$$f(t) = \left[1 - \frac{v^2(t)}{c^2}\right]^{\frac{1}{4}} = K. \tag{18.22}$$

Let us now recall the property of the Brownian motion called self-similarity to the order 1/2 and expressed by the formula $B(ct) = \sqrt{c}B(t)$ where c is any real positive constant—see (21.6) for the relevant proof. From this and from (18.3), one gets at once

$$X(t) = B\left(\int_0^t K^2\, ds\right) = B(K^2 t) = KB(t). \tag{18.23}$$

Thus, the *uniform* proper-time Brownian motion $B(\tau) = X(t)$ equals the *uniform* coordinate-time Brownian motion $B(t)$ multiplied by the *constant* K, which is

$$B(\tau) = \left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{4}} B(t). \tag{18.24}$$

The KL expansion of $B(\tau)$ is, of course, the same as that of $B(t)$ apart from the multiplicative factor K. And the relevant eigenfunctions are just sines.

To provide an example of how the machinery outlined in Section 18.1 actually works, we shall now prove this result, also proved in Section 21.3 (or in [4, p. 280]). From (18.10):

$$\chi(t) = \sqrt{K \int_0^t K\, ds} = K\sqrt{t} \tag{18.25}$$

and

$$\chi'(t) = \frac{K}{2\sqrt{t}}. \tag{18.26}$$

The order $\nu(t)$ of the Bessel functions is then found from (18.9):

$$\begin{aligned} \nu(t) &= \sqrt{-\frac{\chi^3(t)}{f^2(t)} \frac{d}{dt}\left[\frac{\chi'(t)}{f^2(t)}\right]} = \sqrt{-\frac{K^3 t^{\frac{3}{2}}}{K^2} \frac{d}{dt}\left[\frac{1}{2K\sqrt{t}}\right]} \\ &= \sqrt{-Kt^{\frac{3}{2}}\left(-\frac{t^{-\frac{3}{2}}}{4K}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2} \end{aligned} \tag{18.27}$$

where both the time t and the constant K have vanished from the result. Simplifications of this kind (further examples will be given in Sections 18.3 and 12.4) are vital to make the mathematical investigations feasible. Since $\nu = \frac{1}{2}$, the relevant Bessel function is [6, p. 54]

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x. \tag{18.28}$$

Thus, from (18.5), (18.27), and (18.28), the KL expansion follows:

$$\begin{aligned} X(t) &= \sqrt{K \int_0^t K\, ds} \sum_{n=1}^{\infty} Z_n N_n J_{\frac{1}{2}}\left(\gamma_n \frac{\displaystyle\int_0^t K\, ds}{\displaystyle\int_0^T K\, ds}\right) \\ &= K \sum_{n=1}^{\infty} Z_n N_n \sqrt{\frac{2T}{\pi \gamma_n}} \sin\left(\gamma_n \frac{t}{T}\right). \end{aligned} \tag{18.29}$$

In this expression the normalization constants N_n are yet to be found. To this end, we must know the γ_n given by (18.11). That is,

$$\frac{K}{2\sqrt{T}} J_{\frac{1}{2}}(\gamma_n) + K\sqrt{T}\left[\frac{K\gamma_n}{K\int_0^T ds} J'_{\frac{1}{2}}(\gamma_n)\right] = 0 \tag{18.30}$$

or, simplifying,

$$\tfrac{1}{2} J_{\frac{1}{2}}(\gamma_n) + \gamma_n J'_{\frac{1}{2}}(\gamma_n) = 0. \tag{18.31}$$

But this is a special case of the more general Bessel functions formula (see [5, p. 11, entry (54)]:

$$\nu J_\nu(z) + z J'_\nu(z) = z J_{\nu-1}(z) \tag{18.32}$$

so that (18.31) actually amounts to

$$J_{-\frac{1}{2}}(\gamma_n) = 0 \tag{18.33}$$

since $\gamma_n \neq 0$. One now has (see [6, p. 55, entry (6)])

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \tag{18.34}$$

so that (18.33) finally becomes the boundary condition:

$$\cos \gamma_n = 0. \tag{18.35}$$

In this case we find the exact γ_n expression to be

$$\gamma_n = n\pi - \frac{\pi}{2} \quad (n = 1, 2, \ldots). \tag{18.36}$$

Reverting now to the normalization constants N_n, (18.12) yields

$$\begin{aligned} 1 &= N_n^2 \left[\int_0^T K\, ds\right]^2 \int_0^1 x \left[J_{\frac{1}{2}}(\gamma_n x)\right]^2 dx \\ &= N_n^2 K^2 T^2 \frac{2}{\pi \gamma_n} \int_0^1 \sin^2(\gamma_n x)\, dx \\ &= N_n^2 K^2 T^2 \frac{1}{\pi \gamma_n^2} [\gamma_n - \sin\gamma_n \cos\gamma_n] = \frac{N_n^2 K^2 T^2}{\pi \gamma_n} \end{aligned} \tag{18.37}$$

from which

$$N_n = \frac{\sqrt{\gamma_n}\sqrt{\pi}}{KT}. \tag{18.38}$$

As for the eigenvalues λ_n, from (18.13) they are given by

$$\lambda_n = \frac{K^2 T^2}{\gamma_n^2} \tag{18.39}$$

and these are also the variances of the independent Gaussian random variables Z_n. It is interesting to point out that the property

$$\sigma_{cZ}^2 = c^2 \sigma_Z^2 \tag{18.40}$$

and (18.39) yield the following proportionality among the proper-time random variables Z_n and the coordinate-time random variables Z_n^0—corresponding to the case $\nu(t) \equiv \frac{1}{2}$, or, from (18.16), $f(t) \equiv 1$:

$$Z_n = K Z_n^0. \tag{18.41}$$

Thus, the KL expansion of the proper-time Brownian motion is

$$B(\tau) = \sum_{n=1}^{\infty} Z_n \sqrt{\frac{2}{T}} \sin\left(\gamma_n \frac{t}{T}\right) = K \sum_{n=1}^{\infty} Z_n^0 \sqrt{\frac{2}{T}} \sin\left(\gamma_n \frac{t}{T}\right) = KB(t) \tag{18.42}$$

and (18.24) is found once again. In other words, passing from one inertial reference frame to another, the random variables Z_n just change their variance according to (18.41), whereas the time eigenfunctions remain the same. In Section 18.5 total energy will also be discussed.

18.3 DECELERATED MOTION

This and the remaining sections are devoted to the case when the proper time is proportional to a real positive power of the coordinate time, namely

$$\tau = Ct^{2H} \quad (t \geq 0) \tag{18.43}$$

C being a constant that will be determined immediately, and H being a real variable whose range has yet to be found. The factor 2 in the exponent is introduced for convenience. By checking (18.43) against (18.1), differentiating, and taking the square root, one gets

$$f(t) = \sqrt{2HC}\, t^{H-\frac{1}{2}}. \tag{18.44}$$

Inserting this into (18.17), the resulting velocity radical reads

$$v(t) = c\sqrt{1 - (2HC)^2 t^{2(2H-1)}}. \tag{18.45}$$

In order to have a real velocity, the inequality

$$(2HC)^2 t^{2(2H-1)} \leq 1 \tag{18.46}$$

must be valid. Moreover, the initial instant is conventionally zero, and the final instant is T, so that the range of H is necessarily greater than one-half. By setting $t = T$, the constant C is determined so as $v(T) = 0$, and one gets

$$C = \frac{1}{2HT^{2H-1}}. \tag{18.47}$$

One can now understand the physical meaning of the motion we are studying. Initially ($t = 0$) the spaceship is traveling at the speed of light. Then it starts decelerating until it stops at the final instant $t = T$. Actually, if we let H vary, we have a family of curves in the $t, v(t, H)$ plane. But we have to be careful: the tangent to all such curves at $t = 0$ must be *horizontal* in order to preserve the physical reality when the spaceship starts decelerating from c to lower speeds (i.e., there cannot be any sudden "speed jump"). Thus, differentiating (18.45)—with C given by (18.47)—with respect to t and then setting $t = 0$, one discovers that the condition on H given $H > \frac{1}{2}$ must physically be replaced by the stronger condition:

$$4H - 3 > 0 \quad \text{hence } H > \tfrac{3}{4} = 0.75. \tag{18.48}$$

An important special case of $v(t, H)$ occurs when $H = 1$: in fact, $v(t)$ is then the upper-right quarter of an ellipse. One also easily infers that, for $1 < H < \infty$, all $v(t)$ curves lie above this arc of ellipse. In the (physically meaningless) limit case $H \to \infty$ the $v(t, H)$ "curve" would be the upper-right quarter of a rectangle. Figure 18.1 shows this set of $v(t, H)$ curves representing the decelerated motion for different values of H.

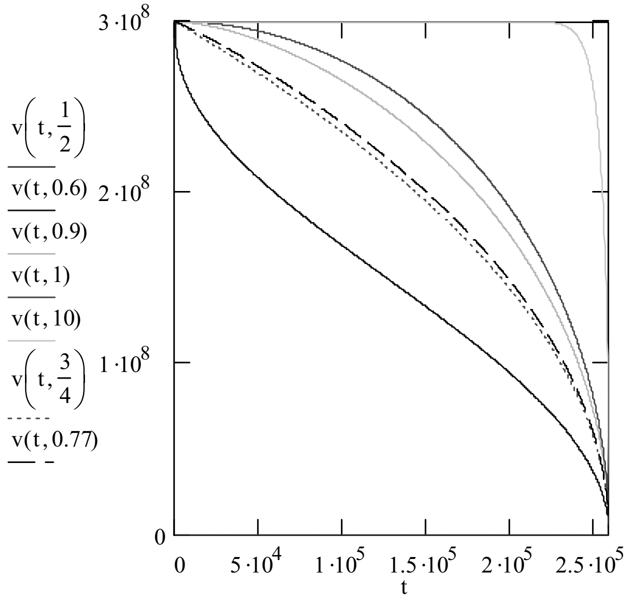


Figure 18.1. Decelerated motion of a relativistic spaceship approaching the Earth at the speed of light c down to speed zero in the finite time interval $0 \le t \le T$. We dubbed this spaceship the *Independence Day* (alien) spaceship. For instance, let $T = 3$ days of coordinate time (i.e., time elapsed on Earth). At the initial instant $t = 0$ (when the deceleration starts) all the curves $v(t, H)$ must have their tangents horizontal (to avoid bumps aboard the spaceship) and that yields the physical constraint: $H > \frac{3}{4} = 0.75$. The above plots show just this fact in a neat, graphic fashion: (1) all solid curves have $H > \frac{3}{4}$ and horizontal tangent at $t = 0$, so they are acceptable; (2) the dividing line is the dash-dotted curve corresponding to $H = \frac{3}{4}$, and one can see that it does not have a horizontal tangent at $t = 0$; (3) all the lower curves (dotted) are not allowed since they don't have a horizontal tangent at $t = 0$.

In conclusion, the function $f(t)$ is defined by the real positive power

$$f(t) = \frac{t^{H-\frac{1}{2}}}{T^{H-\frac{1}{2}}} \quad (0 \le t \le T). \tag{18.49}$$

From (18.17) and (18.49) we see that the velocity $v(t)$ is given by

$$v(t) = c\sqrt{1 - \left(\frac{t^{2H-1}}{T^{2H-1}}\right)^2} \quad (0 \le t \le T) \tag{18.50}$$

One can now understand the physical meaning of the motion we are studying. Initially ($t = 0$) the particle is traveling at the speed of light, then it starts decelerating until it stops at the final instant $t = T$. Actually, (18.50) represents a family of curves on the $(t, \nu(t))$ plane if we let H vary according to (18.48). The particular case $H = \frac{1}{2}$ represents standard Brownian motion. Another important special case of (18.50) occurs when $H = 1$: in fact $v(t)$ is then an ellipse. One also easily infers that, for $\frac{1}{2} < H < 1$ the curve lies below the arc of ellipse, whereas for $1 < H < \infty$ the curve lies above it. In the (physically meaningless) limit case $H \to \infty$ the curve would be half a rectangle.

Let us now turn to the KL expansion of the decelerated Brownian motion

$$X(t) = B\left(\frac{t^{2H}}{2HT^{2H-1}}\right) = \frac{1}{\sqrt{2H}T^{H-\frac{1}{2}}} B(t^{2H}). \tag{18.51}$$

Integrating (18.49), we get

$$\int_0^t f(s)\, ds = \frac{t^{H+\frac{1}{2}}}{(H+\frac{1}{2})T^{H-\frac{1}{2}}}. \tag{18.52}$$

Then, by virtue of (18.49) and (18.52), the function $\chi(t)$ defined by (18.10) reads:

$$\chi(t) = \frac{t^H}{\sqrt{H+\frac{1}{2}}\, T^{H-\frac{1}{2}}} \tag{18.53}$$

thus

$$\chi'(t) = \frac{Ht^{H-1}}{\sqrt{H+\frac{1}{2}}\, T^{H-\frac{1}{2}}}. \tag{18.54}$$

Moreover, from (18.49) and (18.54), one finds the expressions

$$\frac{\chi'(t)}{f^2(t)} = \frac{T^{H-\frac{1}{2}}Ht^{-H}}{\sqrt{H+\frac{1}{2}}} \tag{18.55}$$

and

$$\frac{d}{dt}\left[\frac{\chi'(t)}{f^2(t)}\right] = \frac{T^{H-\frac{1}{2}}H(-H)t^{-H-1}}{\sqrt{H+\frac{1}{2}}} \tag{18.56}$$

and, from (18.49) and (18.53),

$$\frac{\chi^3(t)}{f^2(t)} = \frac{t^{H+1}}{(H+\frac{1}{2})\sqrt{H+\frac{1}{2}}\,T^{H-\frac{1}{2}}}. \tag{18.57}$$

The Bessel functions order can now be found from (18.9), (18.56), and (18.57):

$$\nu = \frac{2H}{2H+1}. \tag{18.58}$$

Note that both the time t and the constant T disappear identically, and the order of the Bessel functions is a constant, rather than a function of the time t. Moreover, by letting $H = \frac{3}{4}$ and $H \to \infty$, respectively, we see that the range of ν is rather limited: $\frac{3}{5} \leq \nu \leq 1$.

Our next task is to find the meaning of the constants γ_n. Upon substituting (18.52), (18.53), and (18.54) into (18.11), along with $\nu'(t) = 0$ one gets, after simplifying any multiplicative factors,

$$\frac{2H}{2H+1} J_\nu(\gamma_n) + \gamma_n J'_\nu(\gamma_n) = 0. \tag{18.59}$$

By virtue of (18.58), (18.59) is equivalent to

$$\nu J_\nu(\gamma_n) + \gamma_n J'_\nu(\gamma_n) = 0. \tag{18.60}$$

Once again the Bessel functions property (18.32) may be applied, and

$$\gamma_n J_{\nu-1}(\gamma_n) = 0. \tag{18.61}$$

Since $\gamma_n \neq 0$,

$$J_{\nu-1}(\gamma_n) = 0. \tag{18.62}$$

Thus, the γ_n are the real positive zeros, arranged in ascending order of magnitude, of the Bessel function of order $\nu - 1$. No formula yielding these zeros explicitly is known. Yet it is possible to find an approximated expression for them by means of the asymptotic formula for $J_\nu(x)$ (see [8, p. 134]).

$$\lim_{x\to\infty} J_\nu(x) = \lim_{x\to\infty} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \tag{18.63}$$

In fact, from (18.58) one first gets

$$\nu - 1 = -\frac{1}{2H+1}. \tag{18.64}$$

Second, (18.62) and (18.64), checked against (18.63), yield

$$0 = J_{\nu-1}(\gamma_n) \approx \sqrt{\frac{2}{\pi\gamma_n}} \cos\left(\gamma_n + \frac{\pi}{2(2H+1)} - \frac{\pi}{4}\right) \tag{18.65}$$

hence

$$\gamma_n + \frac{\pi}{2(2H+1)} - \frac{\pi}{4} \approx n\pi - \frac{\pi}{2} \quad (n = 1, 2, \ldots) \tag{18.66}$$

and finally

$$\gamma_n \approx n\pi - \frac{\pi}{4} - \frac{\pi}{2(2H+1)} \quad (n = 1, 2, \ldots) \tag{18.67}$$

The first 32 approximated γ_n, obtained by means of (18.67), appear in Table 18.2, for various values of $H \geq \frac{1}{2}$. In the Brownian case $H = \frac{1}{2}$ (18.67) is an exact formula, in that it coincides with (18.36). We are reminded that these γ_n give the pace of convergence of the KL expansion, inasmuch as the standard deviations of the Gaussian random variables Z_n depend inversely on the γ_n by virtue of (18.13).

Eventually, the normalization constants N_n follow from (18.12) and (18.52):

$$1 = N_n^2 \frac{T^2}{(H+\frac{1}{2})^2} \int_0^1 x J_\nu^2(\gamma_n x)\, dx. \tag{18.68}$$

This integral is calculated within the framework of the Dini series (see [5, p. 71]) and the result is

$$\int_0^1 x J_\nu^2(\gamma_n x)\, dx = \frac{1}{2\gamma_n^2}[\gamma_n^2 J_\nu'^2(\gamma_n) + (\gamma_n^2 - \nu^2) J_\nu^2(\gamma_n)]. \tag{18.69}$$

This formula, however, may be greatly simplified upon eliminating $\gamma_n J_\nu'(\gamma_n)$ taken from (18.60). In fact, one finds

$$\gamma_n^2 J_\nu'^2(\gamma_n) = \nu^2 J_\nu^2(\gamma_n) \tag{18.70}$$

and (18.68), by virtue of (18.69) and (18.70), becomes

$$1 = N_n^2 \frac{T^2}{(H+\frac{1}{2})^2} \cdot \frac{J_\nu^2(\gamma_n)}{2}. \tag{18.71}$$

Thus

$$N_n = \frac{(H+\frac{1}{2})\sqrt{2}}{T|J_\nu(\gamma_n)|}. \tag{18.72}$$

This is the exact expression of the normalization constants. An approximated expression can be found upon inserting both (18.67) and (18.58) into the approximated

Table 18.2. Approximate values of the constants γ_n.

	$H = 0.5$ Brownian	$H = 0.6$	$H = 0.7$	$H = 0.8$	$H = 0.9$	$H = 1.0$	$H = \infty$
$n = 1$	1.571	1.642	1.702	1.752	1.795	1.833	2.356
$n = 2$	4.712	4.784	4.843	4.894	4.937	4.974	5.498
$n = 3$	7.854	7.925	7.985	8.035	8.078	8.116	8.639
$n = 4$	11.00	11.07	11.13	11.18	11.22	11.26	11.78
$n = 5$	14.14	14.21	14.27	14.32	14.37	14.40	14.92
$n = 6$	17.28	17.36	17.41	17.46	17.50	17.54	18.06
$n = 7$	20.42	20.50	20.55	20.60	20.64	20.68	21.20
$n = 8$	23.56	23.63	23.69	23.74	23.79	23.82	24.35
$n = 9$	26.70	26.77	26.83	26.88	26.93	26.96	27.49
$n = 10$	27.84	27.92	27.98	30.03	30.07	30.11	30.63
$n = 11$	32.99	33.06	33.12	33.17	33.21	33.25	33.77
$n = 12$	36.13	36.20	36.26	36.31	36.35	36.39	36.91
$n = 13$	37.27	37.34	37.40	37.45	37.49	37.53	40.05
$n = 14$	42.41	42.48	42.54	42.59	42.64	42.67	43.20
$n = 15$	45.55	45.62	45.68	45.73	45.78	45.81	46.34
$n = 16$	48.69	48.77	48.83	48.88	48.92	48.96	47.48
$n = 17$	51.84	51.91	51.97	52.02	52.06	52.10	52.62
$n = 18$	54.98	55.05	55.11	55.16	55.20	55.24	55.76
$n = 19$	58.12	58.19	58.25	58.30	58.34	58.38	58.90
$n = 20$	61.26	61.33	61.39	61.44	61.48	61.52	62.05
$n = 21$	64.40	64.47	64.53	64.58	64.63	64.66	65.19
$n = 22$	67.54	67.62	67.67	67.72	67.77	67.81	68.33
$n = 23$	70.69	70.76	70.82	70.87	70.91	70.95	71.47
$n = 24$	73.83	73.90	73.96	74.01	74.05	74.09	74.61
$n = 25$	76.97	77.04	77.10	77.15	77.19	77.23	77.75
$n = 26$	80.11	80.18	80.24	80.29	80.33	80.37	80.90
$n = 27$	83.25	83.32	83.38	83.43	83.48	83.51	84.04
$n = 28$	86.39	86.46	86.52	86.57	86.62	86.66	87.18
$n = 29$	87.53	87.61	87.67	87.72	87.76	87.80	90.32
$n = 30$	92.68	92.75	92.81	92.86	92.90	92.94	93.46
$n = 31$	95.82	95.90	95.95	96.00	96.04	96.08	96.60
$n = 32$	98.96	97.0	97.0	97.1	97.1	97.2	97.75

(18.63) for $J_\nu(\gamma_n)$:

$$|J_\nu(\gamma_n)| \approx \left|\sqrt{\frac{2}{\pi\gamma_n}}\cos\left(n\pi - \frac{\pi}{4} - \frac{\pi}{2(2H+1)} - \frac{\pi 2H}{2(2H+1)} - \frac{\pi}{4}\right)\right|$$

$$\approx \sqrt{\frac{2}{\pi\gamma_n}}|\cos(n\pi - \pi)| \approx \sqrt{\frac{2}{\pi\gamma_n}}. \tag{18.73}$$

By substituting this into (18.72) and using (18.67) for the γ_n, it follows that

$$N_n \approx \frac{\pi}{T}(H + \tfrac{1}{2})\sqrt{n - \frac{1}{4} - \frac{1}{2(2H+1)}}. \tag{18.74}$$

These are the approximated normalization constants.

A similar procedure applies to the eigenvalues λ_n. In fact, from (18.13) and (18.52) we get the exact formula

$$\lambda_n = \frac{T^2}{(H + \frac{1}{2})^2} \cdot \frac{1}{(\gamma_n)^2} \tag{18.75}$$

whereas from (18.75) and (18.67) we get the approximated formula

$$\lambda_n \approx \frac{T^2}{(H + \frac{1}{2})^2} \cdot \frac{1}{\pi^2\left(n - \frac{1}{4} - \frac{1}{2(2H+1)}\right)^2}. \tag{18.76}$$

These are the variances of the independent Gaussian random variables Z_n.

Let us now summarize all the results found in the present section by writing two KL expansions: the exact one

$$X(t) = \frac{\sqrt{2H+1}\,t^H}{T^{H+\frac{1}{2}}} \sum_{n=1}^{\infty} Z_n \frac{1}{|J_\nu(\gamma_n)|} J_\nu\left(\gamma_n \frac{t^{H+\frac{1}{2}}}{T^{H+\frac{1}{2}}}\right) \tag{18.77}$$

and the approximated one

$$X(t) \approx \frac{\sqrt{2H+1}\,t^{\frac{H}{2}-\frac{1}{4}}}{T^{\frac{H}{2}+\frac{1}{4}}} \sum_{n=1}^{\infty} Z_n \cos\left(\gamma_n \frac{t^{H+\frac{1}{2}}}{T^{H+\frac{1}{2}}} - \frac{2H\pi}{2(2H+1)} - \frac{\pi}{4}\right). \tag{18.78}$$

18.4 CHECKING THE KLT OF DECELERATED MOTION BY MATLAB SIMULATIONS

Just look at Figure 18.2.

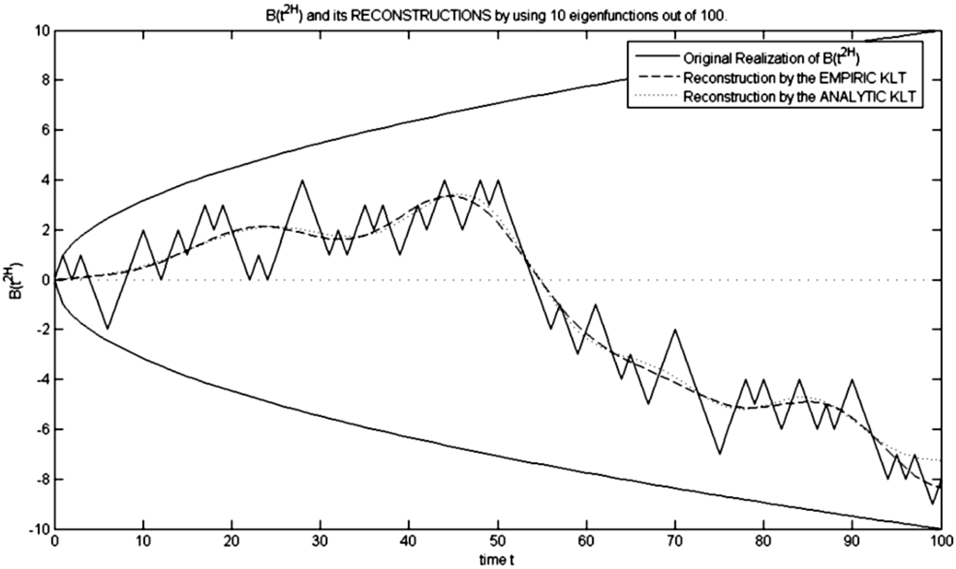


Figure 18.2. The time-rescaled Brownian motion $X(t)$ of (18.78) vs. time t simulated as a random walk over 100 time instants. This $X(t)$ represents the "noisy signal" received on Earth (whence the use of the coordinate time $t =$ Earth time) from a relativistic spaceship approaching the Earth in a decelerated motion, as in the movie *Independence Day*. Next to the "bumpy curve" of $X(t)$, two more "smooth curves" are shown that *interpolate at best* the bumpy $X(t)$. These two curves are the KLT reconstruction of $X(t)$ *by using the first ten eigenfunctions only*. It is important to note that the two smooth curves are *different* in this case because the KLT expansion (18.78) is *approximated*. Actually, it is an approximated KLT expansion because the asymptotic expansion of the Bessel functions (18.63) was used. So, the two curves are different from each other, but both still interpolate $X(t)$ at best. Note that, were we taking into account the full set of 100 KLT eigenfuctions—rather than just 10—then the *empirical* reconstruction would overlap $X(t)$ *exactly*, but the *analytic* reconstruction would not because of the use of the asymptotic expansion (18.63) of the Bessel functions.

18.5 TOTAL ENERGY OF THE NOISY SIGNAL FROM RELATIVISTIC SPACESHIPS IN DECELERATED AND UNIFORM MOTION

A thorough study of the total energy of the noisy signals emitted by relativistic spaceships in decelerated motion (and of the uniform motion, in particular) is allowed by the results obtained in Sections 18.2, 18.3, and 21.10 in Chapter 21.

Our first goal will be to get the characteristic function (i.e., the Fourier transform) of the random variable "total energy", defined by (21.47). In fact, inserting the eigenvalues (18.75) into (21.51), it follows that

$$\Phi_{\varepsilon}(\zeta) = \left[\prod_{n=1}^{\infty}\left(1 - \frac{2iT^2\zeta}{(H+\frac{1}{2})^2\gamma_n^2}\right)\right]^{-\frac{1}{2}}. \tag{18.79}$$

On the other hand,

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right) \tag{18.80}$$

is the infinite product expansion for $J_\nu(z)$ [6, p. 498], and the constants $j_{\nu,n}$ evidently are the real positive zeros of $J_\nu(z)$, arranged in ascending order of magnitude. Then, keeping in mind (18.62), we can let the two infinite products (18.79) and (18.80) coincide by setting

$$\gamma_n = j_{\nu-1,n} \tag{18.81}$$

and

$$z^2 = \frac{2iT^2\zeta}{(H+\frac{1}{2})^2} \quad \text{or} \quad z = T\frac{\sqrt{2i\zeta}}{H+\frac{1}{2}}. \tag{18.82}$$

Solving for $\Phi_\varepsilon(\zeta)$, one gets

$$\Phi_\varepsilon(\zeta) = \frac{1}{\sqrt{\Gamma(\nu)\left[\frac{T\sqrt{2i\zeta}}{2H+1}\right]^{1-\nu} J_{\nu-1}\left(\frac{T\sqrt{2i\zeta}}{H+\frac{1}{2}}\right)}} \tag{18.83}$$

which is the exact expression for the characteristic function of the total energy distribution, ε. An approximated expression can also be derived using the asymptotic expression for the Bessel function (18.63); one then gets

$$\Phi_\varepsilon(\zeta) \approx \frac{1}{\sqrt{\frac{\Gamma(\nu)}{\sqrt{\pi}}\left[\frac{T\sqrt{2i\zeta}}{2H+1}\right]^{\frac{1}{2}-\nu} \cos\left(\frac{T\sqrt{2i\zeta}}{H+\frac{1}{2}} - \frac{\nu\pi}{2} + \frac{\pi}{4}\right)}}. \tag{18.84}$$

In the standard Brownian case $H = \frac{1}{2}$ (hence $\nu = \frac{1}{2}$ and one can apply the formula $\Gamma(\frac{1}{2}) = \sqrt{\pi}$), both (18.83) and (18.84) become

$$\Phi_\varepsilon(\zeta) = \frac{1}{\sqrt{\cos(T\sqrt{2i\zeta})}}. \tag{18.85}$$

This result is due to Cameron and Martin, who published it in 1944 [9].

Our next goal is the computation of all the total energy cumulants, given by (21.56). To this end, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{(\gamma_n)^{2k}} \equiv S_{2k,\nu-1} \equiv \sigma_{\nu-1}^{(k)} \quad (k = 1, 2, \ldots) \tag{18.86}$$

where the notation $S_{2k,\nu-1}$ is used on [5, p. 61], while the notation $\sigma_{\nu-1}^{(k)}$ is used on [6, p. 502]. Then

$$\sum_{k=1}^{\infty} S_{2k,\nu-1} x^{2k-1} = \frac{J_\nu(x)}{2J_{\nu-1}(x)} \tag{18.87}$$

is the power series in x, with coefficients $S_{2k,\nu-1}$, whose proof is given on [5, p. 61].

From the formula that yields any coefficient of a power series, it follows that the coefficients $S_{2k,\nu-1}$ of the power series in x on the left side of (18.87) are given by

$$S_{2k,\nu-1} = \frac{1}{(2k-1)!} \lim_{x\to 0^+} \left[\frac{d^{2k-1}}{dx^{2k-1}} \left(\frac{J_\nu(x)}{2J_{\nu-1}(x)} \right) \right] \tag{18.88}$$

and the sum of the series (18.86) is obtained. Finally, by virtue of (21.56) and (18.88) we conclude that all the cumulants of the total energy are

$$\begin{aligned} K_n &= \frac{2^{n-1} T^{2n}}{(H+\frac{1}{2})^{2n}} \cdot \frac{(n-1)!}{(2n-1)!} \lim_{x\to 0^+} \left[\frac{d^{2n-1}}{dx^{2n-1}} \left(\frac{J_\nu(x)}{2J_{\nu-1}(x)} \right) \right] \\ &= \frac{2^{n-1} T^{2n}}{(H+\frac{1}{2})^{2n}} \cdot (n-1)! \cdot \sigma_{\nu-1}^{(n)} \quad (n = 1, 2, \ldots) \end{aligned} \tag{18.89}$$

where the quantities $\sigma_{\nu-1}^{(1)}$, $\sigma_{\nu-1}^{(2)}$, $\sigma_{\nu-1}^{(3)}$, $\sigma_{\nu-1}^{(4)}$, $\sigma_{\nu-1}^{(5)}$, and $\sigma_{\nu-1}^{(6)}$ appear on [6, p. 502]—ν is to be replaced by H via (18.58).

Having found all the cumulants, we can now derive the expressions of the most interesting statistical parameters of the total energy ε.

(1) Mean value of the total energy:

$$K_1 = E\{\varepsilon\} = \frac{T^2}{2H(2H+1)}. \tag{18.90}$$

(2) Variance of the total energy:

$$K_2 = \sigma_\varepsilon^2 = \frac{T^4}{2H^2(2H+1)(4H+1)}. \tag{18.91}$$

(3) Third total energy cumulant:

$$K_3 = \frac{T^6}{H^3(2H+1)(3H+1)(4H+1)}. \tag{18.92}$$

(4) Fourth total energy cumulant:

$$K_4 = \frac{3(11H+3)T^8}{H^4(2H+1)(3H+1)(4H+1)^2(8H+3)}. \tag{18.93}$$

(5) Skewness of the total energy distribution:

$$\frac{K_3}{(K_3)^{\frac{3}{2}}} = \frac{2^{\frac{3}{2}}\sqrt{2H+1}\sqrt{4H+1}}{3H+1}. \tag{18.94}$$

(6) Kurtosis (or excess) of the total energy distribution:

$$\frac{K_4}{(K_2)^2} = \frac{12(2H+1)(11H+3)}{(3H+1)(8H+3)}. \tag{18.95}$$

Since $H \geq \frac{1}{2}$ we infer from (18.94) that the skewness ranges from $\frac{8}{5}\sqrt{3} = 2.7712813$ (for $H = \frac{1}{2}$) to $\frac{8}{3} = 2.6666667$ for $H \to \infty$. In addition, from (18.95) we find that the kurtosis ranges from $\frac{408}{35} = 11.657143$ for $H = \frac{1}{2}$ to 11 for $H \to \infty$. Therefore, we may conclude that the total energy peak is narrow for any $H \geq \frac{1}{2}$.

The ordinary Brownian motion case of all the previous results is noteworthy, and, relativistically speaking, corresponds to the uniform motion of the moving reference frame with zero velocity (i.e., no motion at all). In fact, by substituting $H = \frac{1}{2}$, $\nu = \frac{1}{2}$ and both (18.28) and (18.34) into (18.89), we find all the Brownian motion total energy cumulants

$$K_n = 2^{n-2} T^{2n} (n-1)! \frac{1}{(2n-1)!} \lim_{x \to 0^+} \left[\frac{d^{2n-1} \tan x}{dx^{2n-1}} \right]. \tag{18.96}$$

Evidently, the last two terms are the $(2n-1)$th coefficient in the MacLaurin expansion of $\tan x$, that reads [5, p. 51]

$$\tan x = \sum_{n=1}^{\infty} \frac{1}{(2n)!} 2^{2n} (2^{2n} - 1)(-1)^{n+1} B_{2n} x^{2n-1}, \tag{18.97}$$

where the B_{2n} are the Bernoulli numbers, a table of which is found, for instance, on [7, p. 810]. Thus, by inserting the coefficients of (18.97) into (18.96), we get all the cumulants of the total energy of standard Brownian motion:

$$K_n = T^{2n} \frac{(n-1)!}{(2n)!} 2^{3n-2} (2^{2n} - 1)(-1)^{n+1} B_{2n}. \tag{18.98}$$

In particular, we have:

(1) mean value of the total energy

$$K_1 = E\{\varepsilon\} = \frac{T^2}{2}; \tag{18.99}$$

(2) variance of the total energy

$$K_2 = \sigma_\varepsilon^2 = \frac{T^4}{3}; \tag{18.100}$$

(3) skewness of the total energy distribution

$$\text{skewness} = \frac{8}{5}\sqrt{3} = 2.7712812921102; \tag{18.101}$$

(4) kurtosis (or excess) of the total energy distribution

$$\text{kurtosis} = \frac{408}{35} = 11.657. \tag{18.102}$$

18.6 *INDEPENDENCE DAY* MOVIE: EXPLOITING THE KLT TO DETECT AN ALIEN SPACESHIP APPROACHING THE EARTH IN DECELERATED MOTION

Everybody remembers the 1996 movie *Independence Day* (see *http://en.wikipedia.org/wiki/Independence_Day_%28film%29*): huge alien spaceships first appear close to Moon and move slowly to prepare for the final attack! It is to be believed, however, that if they move slowly when they are at the Moon distance, they must have moved much, much faster when they were in the open interstellar space in order to cover the vast interstellar distances (please note that here we stick to special relativity only, and do not wish to consider "exotic" mathematical tricks like wormholes, stemming out of general relativity).

In other words, the alien spaceships must have *decelerated* in some way from (say) the speed of light c to zero speed with respect to the Earth. Well, in this section we are going to study the *decelerated* signals emitted by the aliens while they approach the Earth, and work out some equations about the energy of such signals that might help us to dectect an alien invasion much in advance thanks to the KLT developed in this chapter (in the movie *Independence Day*, on the contrary, aliens are already at the Moon distance when humans detect them!).

To adjust our theory to the problem, first consider a trivial Newtonian problem: How long would it take to decelerate from speed c to 0 at the uniform deceleration of just $1g = 9.8\,\mathrm{m/s^{-2}}$? The trivial calculation yieds about 1 year (in Earth time) and the distance at which the deceleration must start is 30,000 AU, or about half a light year (Oort cloud distance) . Should aliens and/or their gadgets withstand decelerations of $2g$, the overall deceleration time would take about half a year, and it should start at the closer distance of 7,600 AU = 0.12 lt-yr from Earth.

Let us now go back to the relativistic decelerated speed $v(t)$ given by (18.50) and consider the radial distance $r(t)$ covered by the spacecraft during the deceleration phase:

$$\frac{dr(t)}{dt} = v(t) = c\sqrt{1 - \left(\frac{t^{2H-1}}{T^{2H-1}}\right)^2}; \tag{18.103}$$

that is

$$R_H(t) = \int_0^{T_H} r(t)\,dt = \int_0^{T_H} c\sqrt{1 - \left(\frac{t^{2H-1}}{T^{2H-1}}\right)^2}\,dt. \tag{18.104}$$

Unfortunately, this integral cannot be computed in a closed form, and we are thus prevented from fully extending our investigation to any value of H larger than $\frac{3}{4}$. We shall thus confine ourselves to the two values $H = \frac{3}{4}$ and $H = 1$, for which one finds

$$R_{\frac{3}{4}}(T) = \int_0^{T_H} r(t)\,dt = \int_0^{T_H} c\sqrt{1 - \frac{t}{T}}\,dt = \tfrac{2}{3}cT_{\frac{3}{4}} = 0.66cT_{\frac{3}{4}} \tag{18.105}$$

and

$$R_1(T) = \int_0^{T_H} r(t)\,dt = \int_0^{T_H} c\sqrt{1-\left(\frac{t}{T}\right)^2}\,dt = \frac{\pi}{4}cT_1 = 0.78cT_1, \quad (18.106)$$

respectively.

Next we are going to focus only on (18.105) because this is the case where the deceleration of the alien spacecraft is "smoothest" (i.e., less sudden).

The total mean energy emitted by the alien spacecraft in the form of electromagnetic waves (= signals + noise) during the time $T_{\frac{3}{4}}$ is given by (18.90) with $H = \frac{3}{4}$; that is

$$K_1 = E\{\varepsilon\} = \frac{T_{\frac{3}{4}}^2}{2H(2H+1)} = \frac{4}{15}T_{\frac{3}{4}}^2 = 0.266T_{\frac{3}{4}}^2. \quad (18.107)$$

The variance of the total energy is given by (18.91) again with $H = \frac{3}{4}$

$$K_2 = \sigma_\varepsilon^2 = \frac{T_{\frac{3}{4}}^4}{2H^2(2H+1)(4H+1)} = \frac{4}{45}T_{\frac{3}{4}}^4 = 0.088T_{\frac{3}{4}}^4. \quad (18.108)$$

Thus, the total mean energy of the electromagnetic waves emitted by the approaching alien spacecraft lies within the range

$$E\{\varepsilon\} \pm \sigma_\varepsilon = \frac{4}{15}T_{\frac{3}{4}}^2 \pm \frac{2}{3\sqrt{5}}T_{\frac{3}{4}}^2 = (0.266 \pm 0.298)T_{\frac{3}{4}}^2. \quad (18.109)$$

This is the "energy bandwidth" upon which any detector of electromagnetic radiation emitted by the alien spacecraft must be built.

The topics discussed in this section were first presented by the author in October 1994 at the *International Astronautical Congress*, held in Jerusalem [10].

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