AA Station Beam Accuracy

AAVP Workshop 2011

Stefan J. Wijnholds

15 December 2011

Station beam requirements

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- field-of-view / station size [1,3]
- sensitivity / aperture efficiency [1,2,3]
- side lobe level [1,3]
- beam accuracy / beam stability [4]

[1] Wijnholds, Bregman & Van Ardenne, Radio Science, 2011

[2] Wijnholds, Grainge, Nijboer & Bregman, URSI, Aug. 2011

[3] Wijnholds, SKA Cal. & Imaging Workshop, Sept. 2011

[4] Wijnholds, 3GC-II Workshop, Sept. 2011

The pivotal question

“What station beam accuracy do we need to achieve $>$ 70 dB dynamic range”

- effective noise [1] consists of
  1. thermal noise
  2. calibration & imaging artefacts
  3. confusion noise
- the first two sources should balanced
- the third should not play a role

[1] Wijnholds & Van der Veen, IEEE JSTSP, Oct. 2008

Setting the scene

Results with SAGEcal with LOFAR HBA [1][2]

- three 6-h observations on 3C196 and NCP
- DDE solutions in $\sim$ 150 directions
- update rate: 20 minutes, 195 KHz
- dynamic range 1:700.000 (3C196)
- noise level 1.2 – 1.5 times thermal noise

[1] Yatawatta, EoR plenary meeting, 29 Nov. 2011

[2] Labropoulos, EoR plenary meeting, 29 Nov. 2011

Imaging

Model based imaging (OMM, ML, LS) regard imaging as an inversion problem:

$$\mathbf{v} = \mathbf{M}\boldsymbol{\sigma}$$

where

	$\mathbf{v}$	$NP^2 \times 1$ visibility vector
	$\mathbf{M}$	$NP^2 \times Q$ measurement model matrix
	$\boldsymbol{\sigma}$	$Q \times 1$ image vector
	N	number of snapshots (correlator dumps)
	P	number of stations
	Q	number of image values

Generic error propagation

$$\frac{\|\Delta \boldsymbol{\sigma}\|}{\|\boldsymbol{\sigma}\|} = \kappa\left(\mathbf{M}\right)\left(\frac{\|\Delta \mathbf{M}\|_2}{\|\mathbf{M}\|_2} + \frac{\|\Delta \mathbf{v}\|}{\|\mathbf{v}\|}\right)$$

in which we recognize three components

1. condition number, noise amplification
2. modeling errors
3. thermal noise

We want

$$\frac{\|\Delta \mathbf{M}\|_2}{\|\mathbf{M}\|_2} \leq \frac{\|\Delta \mathbf{v}\|}{\|\mathbf{v}\|}$$

From 2-norm to Frobenius norm

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$$\|\mathbf{M}\|_2 := \max\left\{\frac{\|\mathbf{Mx}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{C}^Q, \mathbf{x} \neq \mathbf{0}\right\} = \sigma_{\max}$$

is hard to compute, so we use

$$\|\mathbf{M}\|_2 \leq \|\mathbf{M}\|_\mathrm{F} \leq \sqrt{Q}\,\|\mathbf{M}\|_2$$

to reformulate our reqirement to

$$\frac{\|\Delta\mathbf{M}\|_\mathrm{F}}{\|\mathbf{M}\|_\mathrm{F}} \leq \frac{1}{\sqrt{Q}}\frac{\|\Delta\mathbf{v}\|_\mathrm{F}}{\|\mathbf{v}\|_\mathrm{F}}$$

The rigorous approach

$$
\begin{aligned}
\|\Delta \mathbf{M}\|_{\mathrm{F}} &= \sqrt{\sum_{i=1}^{NP^2} \sum_{j=1}^{Q} |\Delta M_{ij}|^2} \\
&= \sqrt{\mathbf{1}^T \mathrm{diag}\left(\mathrm{cov}\left(\mathrm{vec}\left(\mathbf{M}\right)\right)\right)},
\end{aligned}
$$

where

$$
\mathrm{cov}\left(\mathrm{vec}\left(\mathbf{M}\right)\right) = \left(\frac{\partial \mathrm{vec}\left(\mathbf{M}\right)}{\partial \boldsymbol{\theta}^T}\right) \mathrm{cov}\left(\boldsymbol{\theta}\right) \left(\frac{\partial \mathrm{vec}\left(\mathbf{M}\right)}{\partial \boldsymbol{\theta}^T}\right)^T
$$

tedious route, so let's try some clever estimates...

Estimation of $\|\Delta \mathrm{M}\|_{\mathrm{F}}$

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- assume relative error ϵ, i.e., $\Delta M_{ij} = \epsilon \left| M_{ij} \right|$
- we get

$$\|\Delta \mathbf{M}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{NP^2} \sum_{j=1}^{Q} \epsilon^2 \left| M_{ij} \right|^2} = \epsilon \, \|\mathbf{M}\|_{\mathrm{F}}$$

- independent and same relative errors for all stations: $\epsilon = \sqrt{2}\epsilon_{\mathrm{stat}}$

Estimation of $\|\Delta \mathbf{v}\|_\mathrm{F} \,/\, \|\mathbf{v}\|_\mathrm{F}$

- source statistics: $\rho \propto S^\alpha \Leftrightarrow S \propto \rho^{1/\alpha}$
- this implies $\sigma_q = S_\mathrm{max} q^{1/\alpha}$
- homogenous noise: $\Delta\sigma_q = \Delta S_0/\sqrt{B\tau} \;\; \forall\, q$
- assume $\kappa\left(\mathbf{M}\right) = 1$ (DFT matrix)
- in this ideal case, after a little mathematics

$$\frac{\|\Delta \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\|\Delta \boldsymbol{\sigma}\|}{\|\boldsymbol{\sigma}\|} = \frac{\Delta S_0}{S_\mathrm{max}} \cdot \frac{1}{\sqrt{B\tau}} \cdot \frac{\sqrt{Q}}{\sqrt{\sum_{q=1}^{Q} q^{2/\alpha}}}$$

Requirement on beam errors

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$$\epsilon_{\mathrm{stat}} \leq \frac{1}{\sqrt{2}} \cdot \frac{\Delta S_0/\sqrt{B\tau}}{S_{\max}} \cdot \frac{1}{\sqrt{\sum_{q=1}^{Q} q^{2/\alpha}}}$$

Interpretation

- dynamic range after integration (B, τ)
- impact of total observed flux

Example: LOFAR

- SAGEcal: $B = 2 \cdot 10^5$ Hz, $\tau = 1200$ s
- $\Delta S_0 = (k/10^{-26}) \cdot (T_{\rm sys}/A_{\rm eff}) = 1.64$ kJy (24-tile, 150 MHz, [1])
- $S_{\rm max} = 14.5$ Jy ($\rho = 150 S^- 1.5$ [2])
- This gives: $\epsilon_{\rm stat} < 0..27$ %
- This allows for $\epsilon_{\rm stat}\sqrt{24} = 1.3$% error per tile.
- stat. cal.: electronics stable within 1%

[1] Wijnholds & Van Cappellen, IEEE TAP, June 2011

[2] Bregman, NFRA Symposium, April 1999

Forward prediction only for SKA

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- assume 74 dB DR with 100 stations
- $\Delta S_0 / \left(S_{\max} \sqrt{B\tau} \right) = 10^{-5.4}$ per station
- $\alpha = -1.2$ gives $\sqrt{\sum_{q=1}^{10^6} q^{2/\alpha}} \approx 1.45$
- $\epsilon_{\mathrm{stat}} < 1.9 \cdot 10^{-6}$
- **This is not going to happen!**
- **We need to estimate and correct DDEs**

Proposed requirement

- 30 dB DR ensures 200 5σ sources ($\alpha = -1$)
- sufficient for accurate beam prediction (200 directions)
- achievable with $\epsilon_{\mathrm{stat}} < 0.05$ %
- since

$$\mathrm{DR} = \frac{S_{\mathrm{max}}\left(\mathrm{FoV}, A_{\mathrm{eff}}/T_{\mathrm{sys}}, f\right)}{\Delta S_0\left(A_{\mathrm{eff}}/T_{\mathrm{sys}}, f\right)} \cdot \sqrt{B\tau},$$

requirements on electronic stability can be derived from this requirement on beam accuracy

Conclusion & future work

Conclusions

- open loop DR of 30 dB needed
- this should be achievable
- if SKA is calibratable, $>$ 70 dB DR is possible
- calibration approach demonstrated by LOFAR

Future work

- verify analysis by refinement and simulation
- improve source statistics

Alternative derivation

- use

$$\|\Delta\mathbf{M}\|_2 = \sigma_{\max} = \sqrt{\lambda_{\max}\left(\Delta\mathbf{M}^H\Delta\mathbf{M}\right)}$$

- assume $\Delta M \sim \mathcal{N}\left(\mathbf{0}, \epsilon^2\mathbf{I}\right)$
- then [1]

$$\lim_{P,Q\to\infty} \lambda_{\max} = 2\epsilon^2 Q\left(1 + \sqrt{Q/P^2}\right)^2 \text{ for } \frac{Q}{P^2} \in [0,1]$$

- assume $Q \approx P^2$, so $\lambda_{\max} = 8\epsilon^2 Q$.

[1] Edelman, Ph.D. Thesis, MIT, 1999

Estimation of $\|\Delta\mathbf{M}\|_2 \,/\, \|\mathbf{M}\|_2$

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- we found $\|\Delta\mathbf{M}\|_2 = \sqrt{8\epsilon^2 Q} = \epsilon\sqrt{8Q}$
- independent and same relative errors for all stations: $\epsilon = \sqrt{2}\epsilon_{\text{stat}}$
- assume $\mathbf{M}$ resembles the $Q \times Q$ DFT matrix
- this gives $\|\mathbf{M}\|_2 = \sqrt{Q}$
- therefore

$$\frac{\|\Delta\mathbf{M}\|_2}{\|\mathbf{M}\|_2} = \frac{\epsilon_{\text{stat}}\sqrt{16Q}}{\sqrt{Q}} = 4\epsilon_{\text{stat}}$$

Estimation of $\|\Delta\mathbf{v}\|_{\mathrm{F}} \,/\, \|\mathbf{v}\|_{\mathrm{F}}$

- note $\operatorname{var}(\mathbf{v}) = \frac{1}{\sqrt{B\tau}}\mathbf{v} \odot \overline{\mathbf{v}}$
- hence

$$\|\Delta\mathbf{v}\| = \sqrt{\sum_{i=1}^{NP^2} \sigma\left(v_i\right)^2} = \sqrt{\sum_{i=1}^{NP^2} \frac{\left|v_i\right|^2}{B\tau}} = \frac{\|\mathbf{v}\|}{\sqrt{B\tau}}$$

- thus $\|\Delta\mathbf{v}\| \,/\, \|\mathbf{v}\| = 1/\sqrt{B\tau}$
- deconvolution of system noise is not useful
- therefore...

Requirement on beam errors

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- source statistics: $\rho \propto S^{\alpha} \Leftrightarrow S \propto \rho^{1/\alpha}$
- this implies $\sigma_q = S_{\max} q^{1/\alpha}$
- putting it all together:

$$\epsilon_{\text{stat}} \leq \frac{1}{4} \cdot \frac{\Delta S_0/\sqrt{B\tau}}{S_{\max}} \cdot \frac{1}{\sqrt{\sum_{q=1}^{Q} q^{2/\alpha}}}$$

- dynamic range after integration (B, τ)
- impact of total observed flux