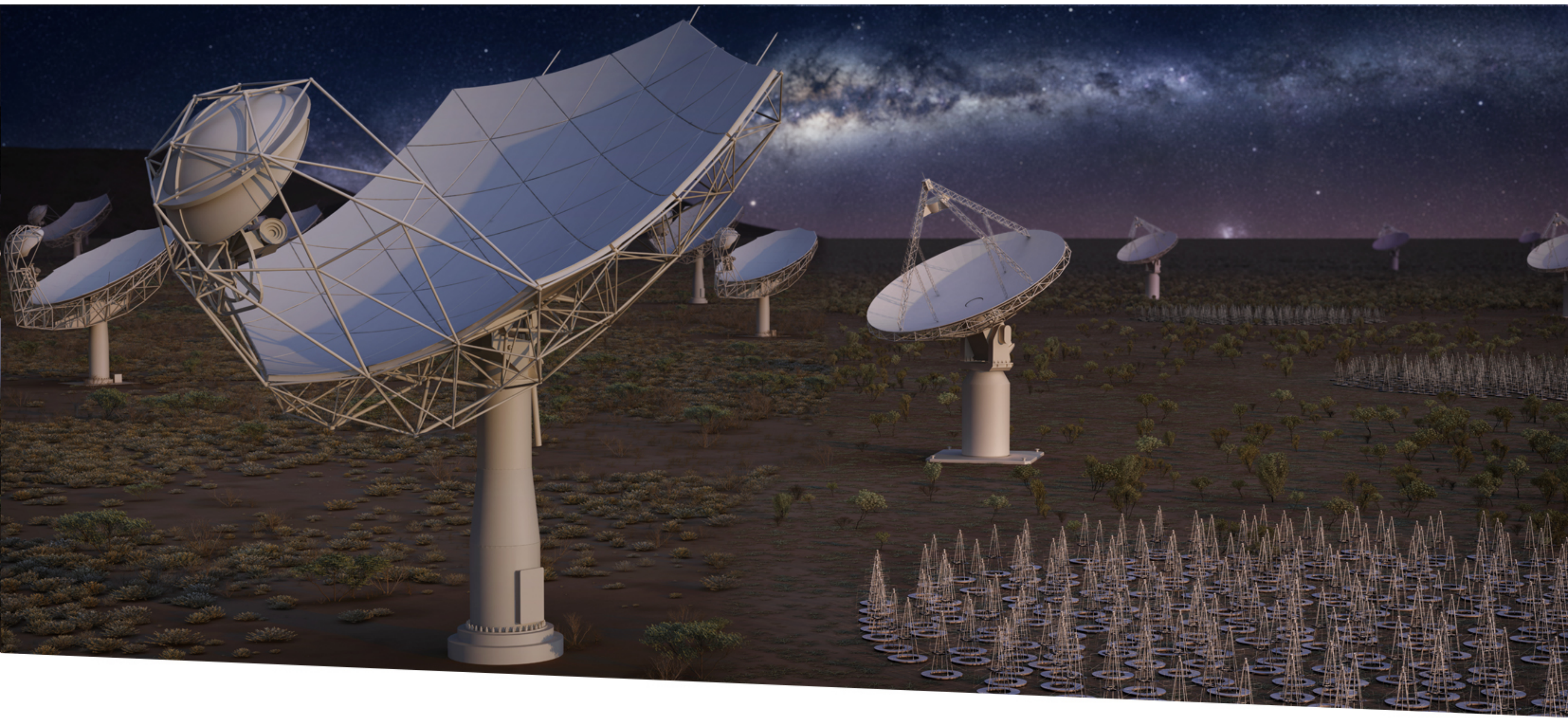
# Fundamentals of Interferometry



SQUARE KILOMETRE ARRAY

Exploring the Universe with the world's largest radio telescope

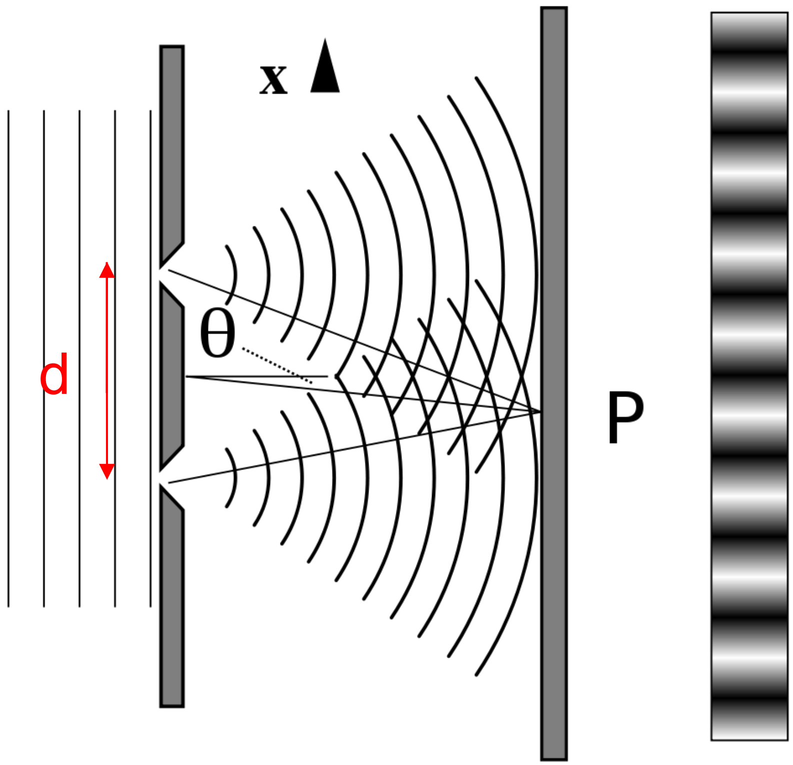
**Robert Laing**

ASTRON, Oct 16 2017

# Objectives

- A more formal approach to radio interferometry using coherence functions
  - A complementary way of looking at the technique
  - Be clear about simplifying assumptions
- Relaxing the assumptions
- How does a radio interferometer work?
  - Follow the signal path
  - Technologies for different frequency ranges

# Young's Slit Experiment



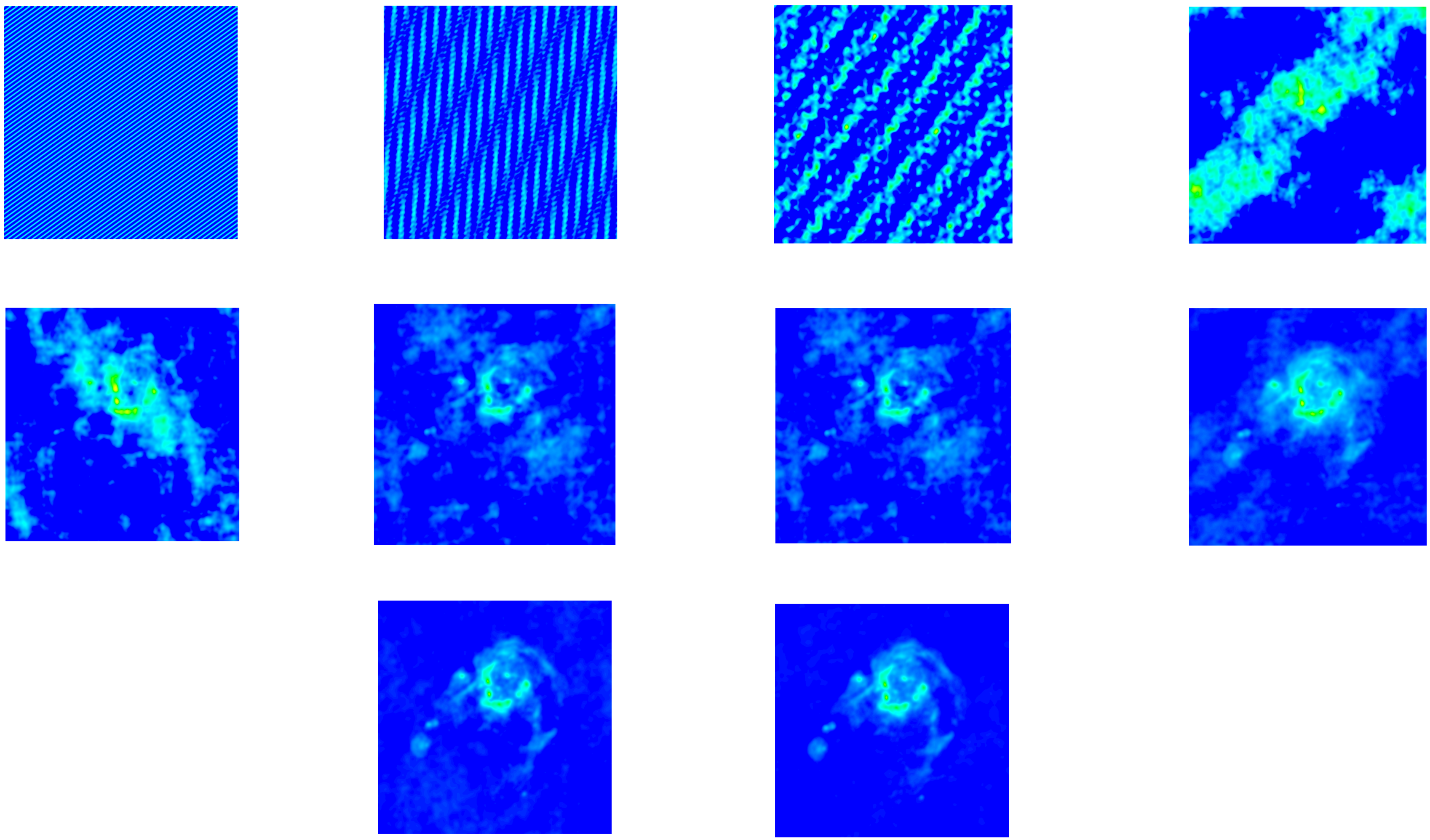

Angular spacing of fringes = λ/d

Familiar from optics

Essentially the way that astronomical interferometers work at optical and infrared wavelengths (e.g. VLTI)

"Direct detection"

# Build up an image from many slits



# But this is not how radio interferometers work in practice ….

- The two techniques are closely related, and it often helps to think of images as built up of sinusoidal "fringes"
- But radio interferometers collect radiation ("antenna"), turn it into a digital signal ("receiver") and generate the interference pattern in a special-purpose computer ("correlator")
- How does this work?
- I find it easiest to start with the concept of the mutual coherence (or correlation) of the radio signal received from the same object at two different places
- No proofs, but I will try to state the simplifying assumptions clearly and return to them later.

# The ideal interferometer (1)

- Astrophysical source, location **R**, generates a time-varying electric field **E**(**R**,t). EM wave propagates to us at point **r**.
- In frequency components: **E**(**R**,t) = $\int \mathbf{E}_\nu(\mathbf{R}) \exp(2\pi i \nu t) d\nu$

The coefficients $\mathbf{E}_\nu(\mathbf{R})$ are complex vectors (amplitude and phase; two polarizations)

- Simplification 1: radiation is monochromatic

$$\mathbf{E}_\nu(\mathbf{r}) = \iiint P_\nu(\mathbf{R},\mathbf{r})\mathbf{E}_\nu(\mathbf{R})\, dx\, dy\, dz$$

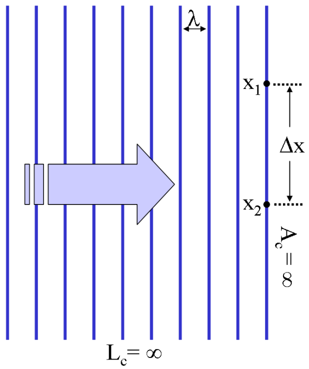
where $P_\nu(\mathbf{R},\mathbf{r})$ is the propagator

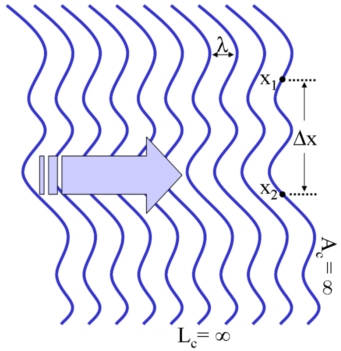
- Simplification 2: scalar field (ignore polarization for now)
- Simplification 3: sources are all very far away
- This is equivalent to having all sources at a fixed distance – there is no depth information

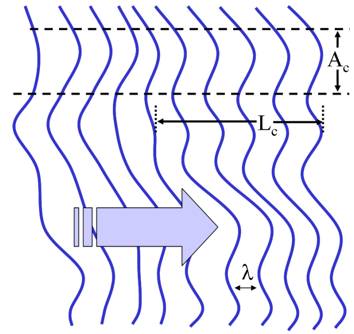
# The ideal interferometer (2)

- Simplification 4: space between us and the source is empty
- In this case, the propagator is quite simple (Huygens' Principle), so:
  $E_\nu(\mathbf{r}) = \int E_\nu(\mathbf{R})\{\exp[2\pi i|\mathbf{R}-\mathbf{r}|/c]/|\mathbf{R}-\mathbf{r}|\}\, dA$
  (dA is the element of area at distance $|\mathbf{R}|$)
- We can measure is the correlation of the field at two different observing locations. This is
  $$C_\nu(\mathbf{r}_1,\mathbf{r}_2) = \langle E_\nu(\mathbf{r}_1)E^*_\nu(\mathbf{r}_2)\rangle$$
  where <> denotes an expectation value and * means complex conjugation.
- Simplification 5: radiation from astronomical objects is not spatially coherent ('random noise').
  $$\langle E_\nu(\mathbf{R}_1)E^*_\nu(\mathbf{R}_2)\rangle = 0 \text{ unless } \mathbf{R}_1 = \mathbf{R}_2$$

# Spatial and Temporal Coherence



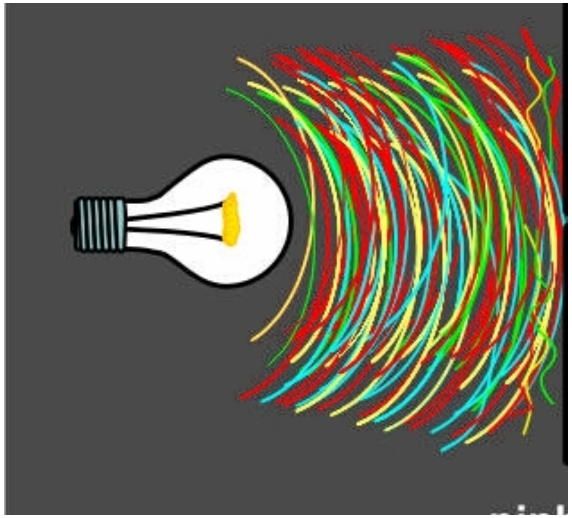




Plane

Varying profile

Spatially coherent

Partially coherent



Most radio sources, like light bulbs are broad-band, incoherent emitters

# Coherent ducks

# The ideal interferometer (3)

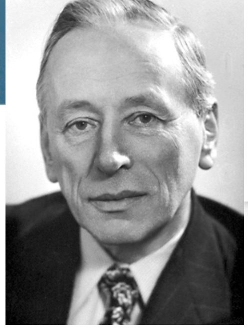
- Now write $\mathbf{s} = \mathbf{R}/|\mathbf{R}|$ and $I_\nu(\mathbf{s}) = |\mathbf{R}|^2 \langle |E_\nu(\mathbf{s})|^2 \rangle$ (the observed intensity). Using the approximation of large distance to the source again:
  $C_\nu(\mathbf{r}_1,\mathbf{r}_2) = \int I_\nu(\mathbf{s}) \exp[-2\pi i \nu \mathbf{s}.(\mathbf{r}_1-\mathbf{r}_2)/c] \, d\Omega$
  ($d\Omega$ is an element of solid angle)
- $C_\nu(\mathbf{r}_1,\mathbf{r}_2)$, the spatial coherence function, depends only on separation $\mathbf{r}_1-\mathbf{r}_2$, so we can keep one point fixed and move the other around.
- It is a complex function, with real and imaginary parts, or an amplitude and phase.

An interferometer is a device for measuring the spatial coherence function

# u,v,w coordinates

- We use a coordinate system (u,v,w), where w is along a reference direction to the phase centre and (u,v) are in the orthogonal plane, with u East-West and v North-South (the u-v plane)
- In this system:
  - Baseline vector between antennas **b** = $(\lambda u, \lambda v, \lambda w)$. Measured in wavelengths.
  - Unit vector to the phase centre $\mathbf{s}_0 = (0,0,1)$
  - Unit vector to some point in the field **s** = (l,m,n), with $l^2+m^2+n^2=1$.

# The Fourier Relation



- Simplification 6: receiving elements have no direction dependence
- Simplification 7: all sources are in a small patch of sky
- Simplification 8: we can measure at all values of $\mathbf{r}_1$-$\mathbf{r}_2$ and at all times
- Choose coordinate system so that the phase tracking centre has $s_0 = (0,0,1)$ as in previous slide.
- $C(\mathbf{r}_1,\mathbf{r}_2) = \exp(-2\pi i w) V'_\nu(u,v)$
- $V'_\nu(u,v) = \iint I_\nu(l,m) \exp[-2\pi i(ul+vm)]\, dl\, dm$
- This is a Fourier transform relation between the modified complex visibility $V'_\nu$ (the spatial coherence function with separations expressed in wavelengths) and the intensity $I_\nu(l,m)$
- “The Fourier Transform of the spatial coherence function of an incoherent source is equal to its complex visibility”: the van Cittert – Zernike Theorem.

# Fourier Inversion

- This relation can be inverted to get the intensity distribution, which is what we want:

  $I_\nu(l,m) = \iint V'_\nu(u,v) \exp[2\pi i(ul+vm)]\, du\, dv$
- This is the fundamental equation of synthesis imaging.
- Interferometrists love to talk about the (u,v) plane. Remember that u, v (and w) are measured in wavelengths.
- The vector $\mathbf{b} = (u,v,w) = (\mathbf{r}_1-\mathbf{r}_2)/\lambda$ is the baseline

# Simplifications

1. Radiation is monochromatic
2. Electromagnetic radiation is a scalar field
3. Sources are all very far away
4. Space between us and the sources is empty
5. Radiation is not spatially coherent
6. Receiving elements have no direction dependence
7. All sources are in a small patch of sky
8. We can measure all baselines at all times

False Sometimes true Almost always true

# Simplification 1

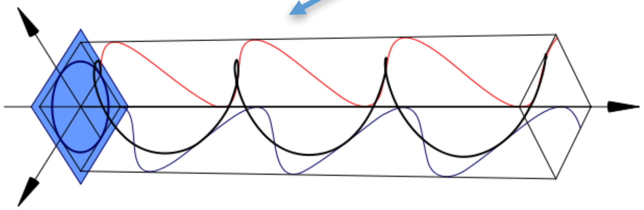
- Radiation is monochromatic
  - We observe wide bands both for spectroscopy (HI, molecular lines) and for sensitive continuum imaging, so we need to get round this restriction.
  - In fact, we can easily divide the band into multiple spectral channels (details later)
  - There are imaging restrictions if the individual channels are too wide for the field size – see imaging lectures.
    - Usable field of view < $(\Delta\nu/\nu_0)(l^2+m^2)^{1/2}$.
    - Not usually an issue for modern instruments, which have large numbers of channels

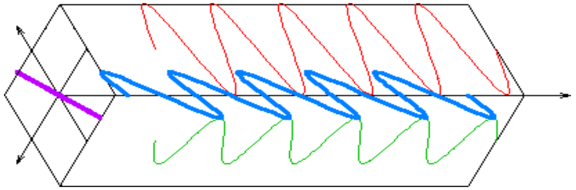
# Simplification 2

- Radiation field is a scalar quantity
  - The field is actually a vector and we are interested in both components (i.e. its polarization).
  - This makes no difference to the analysis as long as we measure two states of polarization (e.g. right and left circular, or crossed linear) and account for the coupling between states.
  - Use the **measurement equation** formalism for this (calibration and polarization lectures).

# Polarization

- Want to image Stokes parameters:
  - I (total intensity)
  - Q, U (linear)
  - V (circular)
- Resolve into two (nominally orthogonal) polarization states, either right and left circular or crossed linear.





# Simplifications 3 and 4

- Sources are all a long way away
  - Strictly speaking, in the far field of the interferometer, so that the distance is $>D^2/\lambda$, where D is the interferometer baseline
  - True except in the extreme case of very long baseline observations of solar-system objects
- Radiation is not spatially coherent
  - Generally true, even if the radiation mechanism is itself coherent (masers, pulsars)
  - May become detectable in observations with very high spectral and spatial resolution
  - Coherence can be produced by scattering, since signals from the **same** location in a sources are spatially coherent, but travel by **different** paths through interstellar or interplanetary medium

# Simplifications 5 and 6

- Space between us and the source is empty
- The receiving elements have no direction-dependence
- Closely related and not true in general. Examples:
  - Interstellar or interplanetary scattering
  - Tropospheric and (especially) ionospheric fluctuations which lead to path/phase and amplitude errors, sometimes seriously direction-dependent
  - Ionospheric Faraday rotation, which changes the plane of polarization
  - High-frequency antennas are highly directional by design
- Standard calibration deals with the case that there is no direction-dependence (i.e each antenna has a single, time-variable complex gain)
- Direction dependence is becoming more important, especially for low frequencies and wide fields.

# Special case: primary beam correction

- If the response of the antenna is direction-dependent, then we are measuring

  $I_v(l,m)\ D_{1v}(l,m)D^*_{2v}(l,m)$ instead of $I_v(l,m)$ (ignore polarization for now)
- An easier case is when the antennas all have the same response

  $A_v(l,m) = |D_v(l,m)|^2$
- $V'_v(u,v) = \iint A_v(l,m) I_v(l,m) \exp[-2\pi i(ul+vm)]\, dl\, dm$
- We just make the standard Fourier inversion and then divide by the **primary beam** $A_v(l,m)$
- $I_v(l,m) = A_v(l,m)^{-1} \iint V'_v(u,v) \exp[2\pi i(ul+vm)]\, du\, dv$

# Simplification 7

- The field of view is small
  - (or antennas are in a single plane)
- Not always true. If not:
  - Basic imaging equation becomes:

$$V_\nu(u,v,w) = \iint I_\nu(l,m) \{\exp[-2\pi i(ul+vm+(1-l^2-m^2)^{1/2}w)]/(1-l^2-m^2)^{1/2}\} \, dl \, dm$$

  - No longer a 2D Fourier transform, so analysis becomes more complicated (the "w term")
  - Map individual small fields ("facets") and combine later, or
  - w-projection
  - See imaging lectures

# Simplification 8

- We can measure the coherence function for any spacing and time.
- Very wrong!
  - We have a number of antennas at fixed locations on the Earth (or in orbit around it)
  - The Earth rotates
  - We make many (usually) short integrations over extended periods, sometimes in separate observations
  - So effectively we sample at discrete u, v (and w) positions.
  - Implicitly assume that the source does not vary
    - Often a problem when combining observations take over a long time period; some sources vary much faster (e.g. the Sun)
  - Also assume that each integration (time average to get the coherence function) is of infinitesimal duration.

# Simplification 8 (continued)

- In 2D, this measurement process can be described by a sampling function $S(u,v)$ which is a delta function where we have taken data and zero elsewhere.
- $I^D_\nu(l,m) = \iint V_\nu(u,v)\, S(u,v) \exp[2\pi i(ul+vm)]\, du\, dv$ is the **dirty image**, which is the Fourier transform of the **sampled** visibility data.
- $I^D_\nu(l,m) = I_\nu(l,m) \otimes B(l,m)$, where the $\otimes$ denotes convolution and
  $B(l,m) = \iint S(u,v) \exp[2\pi i(ul+vm)]\, du\, dv$ is the **dirty beam**
- The process of getting from $I^D_\nu(l,m)$ to $I_\nu(l,m)$ is **deconvolution** (examples in other lectures).
- However, perhaps better to pose the problem in a different way: what model brightness distribution $I_\nu(l,m)$ gives the best fit to the measured visibilities and how well is this model constrained?

# How to build an array

- Antennas
- Receivers
- Down-conversion
- Measuring the correlations
- Spectral channels
- Calibration
- Imaging

Signal Path

What are the key design parameters?

# Resolution and field

d = baseline D = antenna diameter

- Science ➔ Wavelength, λ
- Then:
  - Resolution /rad: $\approx \lambda/d_{max}$
  - Maximum observable scale /rad: $\approx \lambda/d_{min}$
  - Primary beam/rad: $\approx \lambda/D$
- Good coverage of the u-v plane (many antennas, Earth rotation) allows high-quality imaging.
- Sources with all brightness on scales $>\lambda/d_{min}$ are resolved out.
- Sources with all brightness on scales $< \lambda/d_{max}$ look like points

# Noise

$$S_{rms} = \frac{2kT_{sys}}{A_{eff}\sqrt{N_A(N_A - 1)t_{int}\Delta v}}$$

- RMS noise level $S_{rms}$
  - $T_{sys}$ is the system temperature, $A_{eff}$ is the effective area of the antennas, $N_A$ is the number of antennas, Δv is the bandwidth, $t_{int}$ is the integration time and k is Boltzmann's constant
- For good sensitivity, you need low $T_{sys}$ (receivers), large $A_{eff}$ (big, accurate antennas), large $N_A$ (many antennas), long integrations ($t_{int}$) and, for continuum, large bandwidth Δv.
- Best $T_{rec}$ typically a few to ~30K from 1 – 100 GHz, ~100 K at 700 GHz. Atmosphere dominates $T_{sys}$ at high frequencies; foregrounds at low frequencies

# Antennas collect radiation

- Specification, design and cost are frequency-dependent
  - High-frequency: steerable dishes (5 – 100 m diameter)
  - Low-frequency: fixed dipoles, yagis, ....
  - Ruze formula efficiency = $\exp[-(4\pi\sigma/\lambda)^2]$
  - Surface rms error $\sigma < \lambda/20$
  - sub-mm antennas are challenging (surface rms <25 μm for 12m ALMA antennas); offset pointing <0.6 arcsec rms

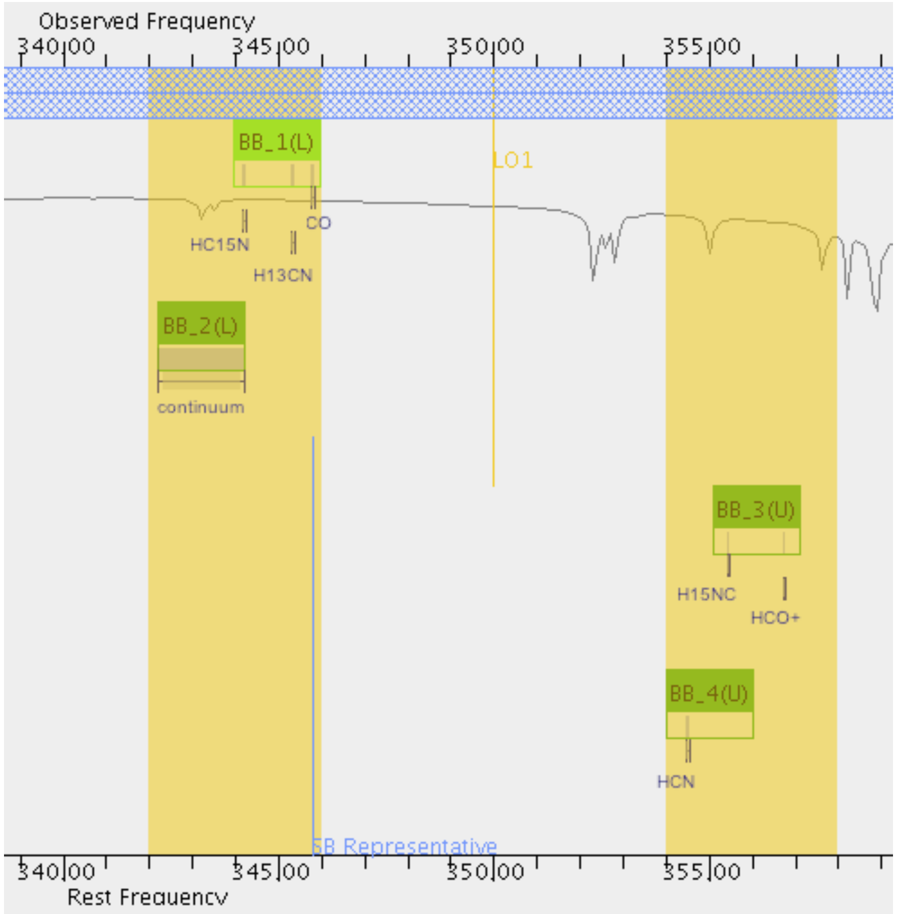
High frequency (ALMA)

Low frequency (LOFAR)

# Receivers

- Detect radiation
- Cryogenically cooled for low noise (at high frequencies)
- Normally detect two polarization states
  - Separate optically
- Optionally, in various combinations:
  - Amplify RF signal
  - Then either:
    - digitize directly (possible up to ~10's GHz) or
    - **mix** with phase-stable local oscillator signal to make intermediate frequency (IF) → two sidebands (one or both used) → digitize
    - a **mixer** is a device with a non-linear voltage response that outputs a signal at the difference frequency
- Digitization typically 3 – 8 bit
- Send to correlator

# Sidebands




ALMA example

LO frequency = 350 GHz
Upper sideband 354-358 GHz
Lower sideband 342-346 GHz

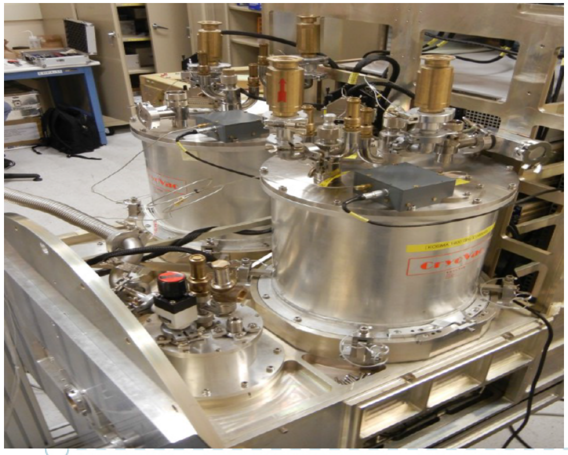
Either separate and keep both sidebands or filter out one.

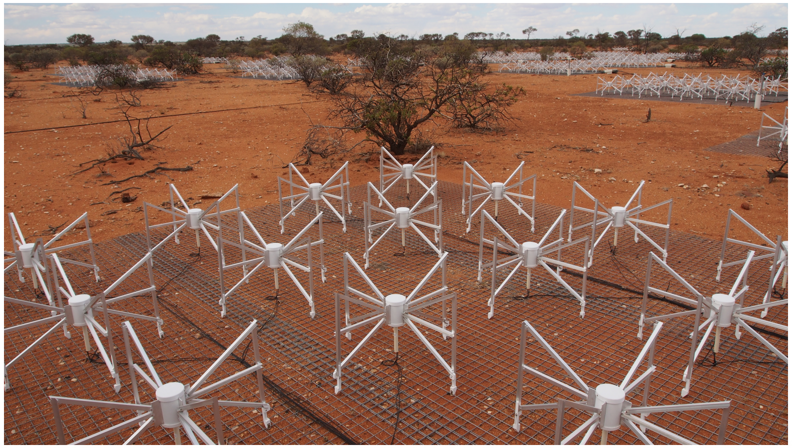
# Time and Frequency Distribution

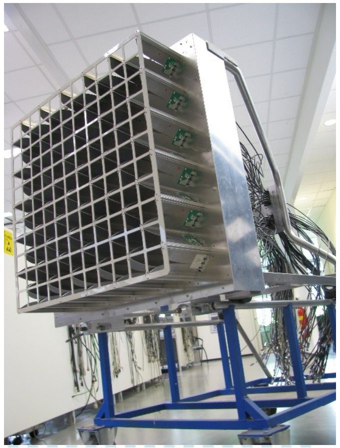
- These days, often done over fibre using optical analogue signal
- Master frequency standard (e.g. H maser)
- Must be phase-stable – round trip measurement
- Slave local oscillators at antennas
  - Multiply input frequency
  - Change frequency within tuning range

# Spatial multiplexing

- Focal-plane arrays (multiple receivers in the focal plane of a reflector antenna).
- Phased arrays with multiple beams from fixed antenna elements (“aperture arrays") e.g. LOFAR, MWA
  - a phased array is an array of antennas from which the signals are combined with appropriate amplitudes and phases to reinforce the response in a given direction and suppress it elsewhere
- Hybrid approach: phased array feeds (= phased arrays in the focal plane of a dish antenna, e.g. APERTIF).



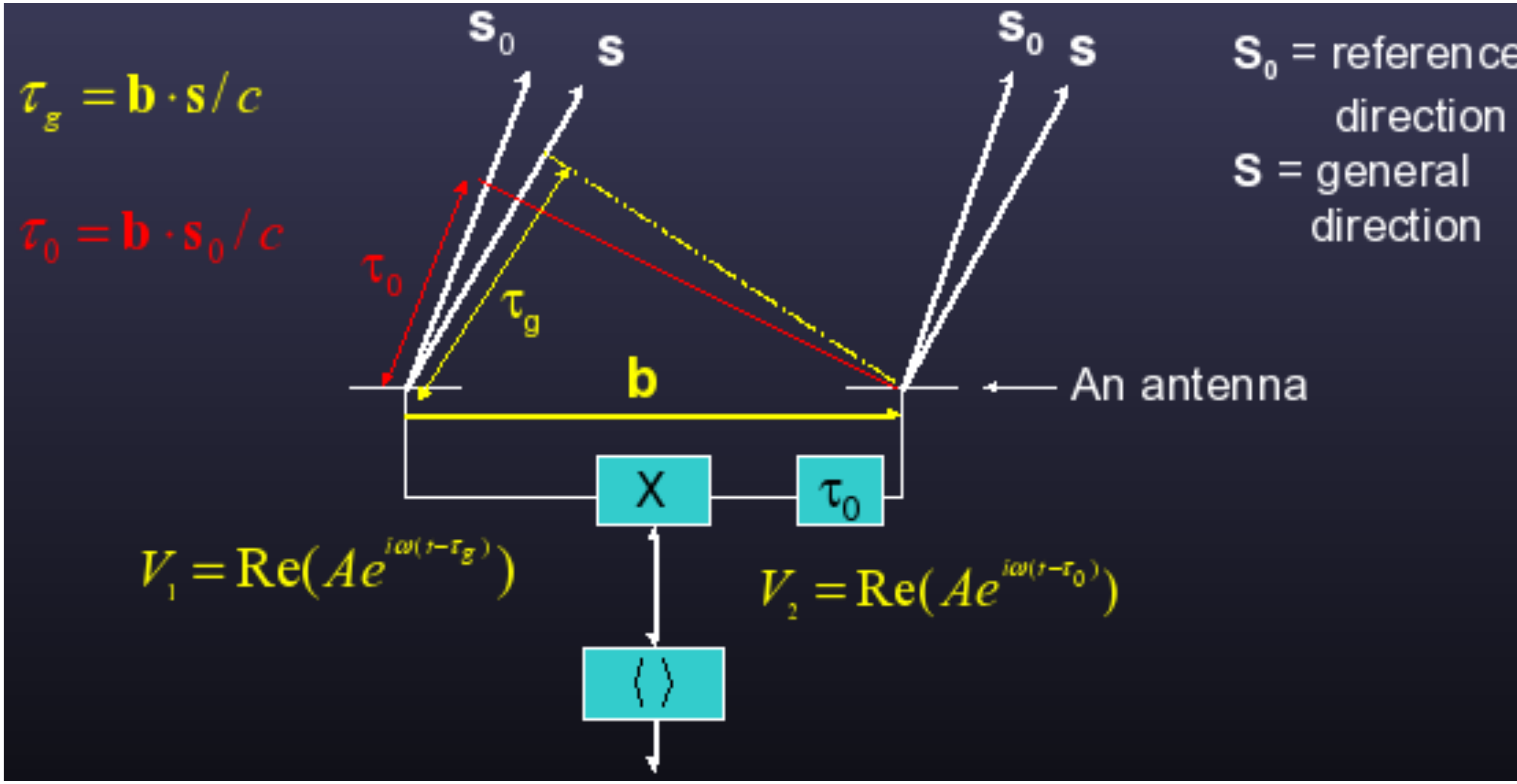




# Delay

- An important quantity in interferometry is the time delay in arrival of a wavefront (or signal) at two different locations, or simply the delay, $\tau$.
- This directly affects our ability to calculate the coherence function
- Examples:
  - Constant (“cable”) delay in waveguide or electronics
  - geometrical delay (next slide)
  - propagation delay through the atmosphere
- Aim to calibrate and remove all of these accurately
- Phase varies linearly with frequency for a constant delay
  - $\Delta\varphi = 2\pi\tau\Delta\nu$
  - Characteristic signature

# Delay Tracking

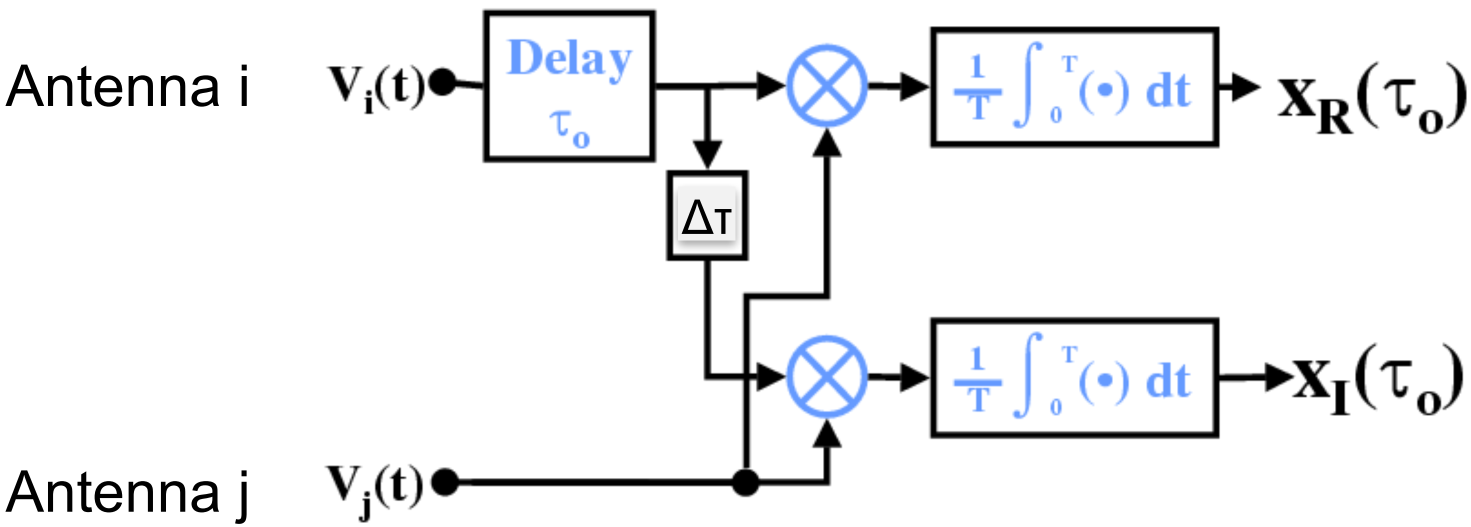



The geometrical delay $\tau_0$ for the delay tracking centre can be calculated accurately from antenna position + Earth rotation model.

Works exactly only for the delay tracking centre. Maximum averaging time is a function of angle from this direction.

# What does a correlator do?

Takes digitized signals from individual antennas; calculates complex visibilities for each baseline




Real

+

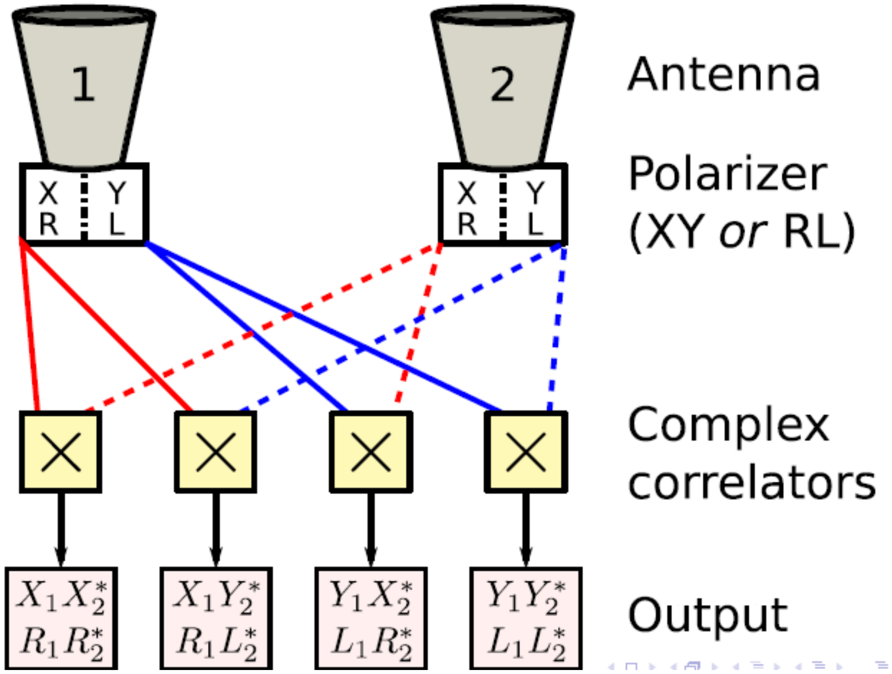
Imaginary

- $x_R = x_{ij}(\tau_0)$
- $x_I = x_{ij}(\tau_0 + \Delta\tau)$, **with** $\Delta\tau = 1/\,(4\nu_0)\ (\Delta\phi = 90°)$.

or amplitude
And phase

# Stokes Parameters and Visibilities

- This assumes the perfect case:
  - no rotation on the sky (not true for most arrays, but can be calculated and corrected)
  - perfect system
- Circular basis
  - $V_I = V_{RR} + V_{LL}$
  - $V_Q = V_{RL} + V_{LR}$
  - $V_U = i(V_{RL} - V_{LR})$
  - $V_V = V_{RR} – V_{LL}$
- Linear basis
  - $V_I = V_{XX} + V_{YY}$
  - $V_Q = V_{XX} - V_{YY}$
  - $V_U = V_{XY} + V_{YX}$
  - $V_V = i(V_{XY} - V_{YX})$



# Spectroscopy

- We make multiple channels by correlating with different values of **lag**, $\tau_0$. This is a delay introduced into the signal from one antenna with respect to another as in the previous slide. For each quasi-monochromatic frequency channel, a lag $\tau$ is equivalent to a phase shift $2\pi\tau\nu$, i.e.

  $$V(u,v,\tau) = \int V(u,v,\nu)\exp(2\pi i\tau\nu)\, d\nu$$

- This is another Fourier transform relation with complementary variables $\nu$ and $\tau$, and can be inverted to extract the desired visibility as a function of frequency.
- In practice, we do this digitally, in finite frequency channels:

  $$V(u,v,j\Delta\nu) = \Sigma_k V(u,v,k\Delta\tau)\exp(-2\pi ijk\Delta\nu\Delta\tau)$$

- Each spectral channel can then be imaged (and deconvolved) individually. The final product is a **data cube**, regularly gridded in two spatial and one spectral coordinate.

I have described an "XF" correlator. The Fourier Transform step can be done first ("FX").

# Calibration Overview

- What we now have is the output from a correlator, which contains signatures from at least:
  - the Earth's atmosphere
  - antennas and optical components
  - receivers, filters, amplifiers
  - digital electronics
  - leakage between polarization states
- What we want is a set of perfect visibilities $V_\nu(u,v,w)$, on an absolute amplitude scale, measured for exactly known baseline vectors (u,v,w), for a set of frequencies, ν, in full polarization.
- Basic idea:
  - Apply a priori calibrations
  - Measure gains for known calibration sources as functions of time and frequency; interpolate to target
  - Refine the calibration on the target

# A priori and on-line calibrations

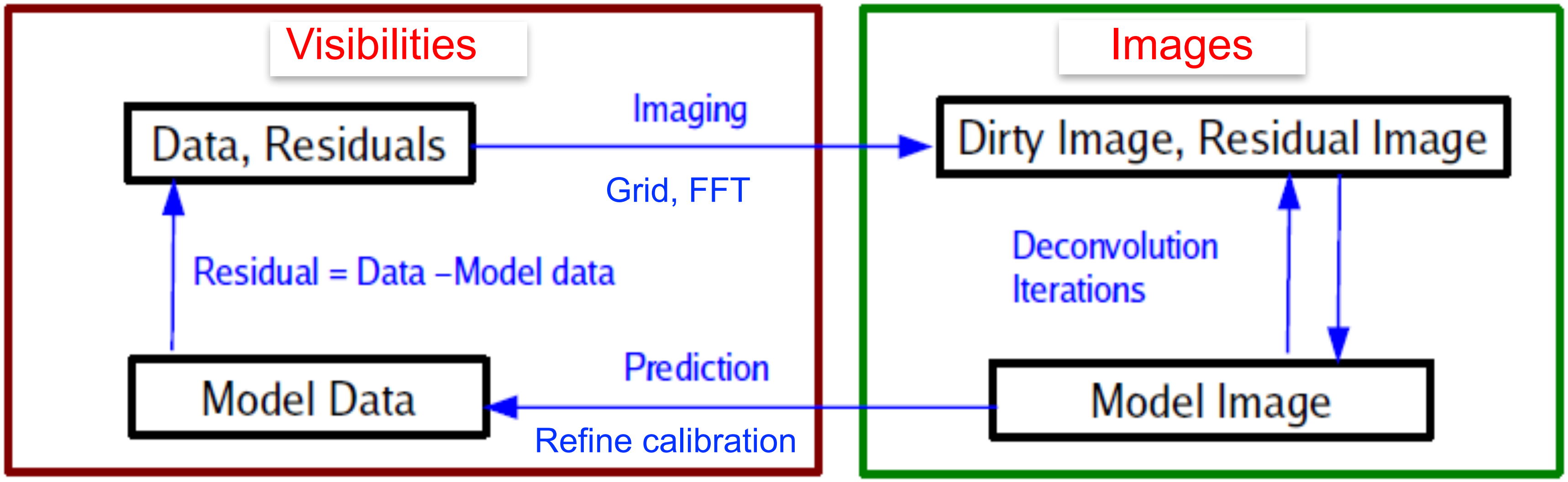
- Array calibrations
  - Antenna pointing and focus model (absolute or offset)
  - Beam forming (aperture arrays and phased-array feeds)
  - Correlator model (antenna locations, Earth orientation and rate, clock, Instrumental delays)
  - Known atmospheric effects (refraction and delay; ionospheric Faraday rotation…)
- On-line calibrations (refinements to array calibrations)
  - Reference pointing and focus
  - Attenuator settings
- On-line flags
  - Antennas not on source
  - Receiver broken
  - Radio frequency interference
- Measured/derived from simulations/.. and applied post hoc
  - Antenna gain curve/voltage pattern as function of elevation, temperature, frequency
  - Receiver non-linearity, quantization
  - Noise diode or load calibration to measure gain variations
  - Delay/attenuation due to varying atmospheric water vapour

Needed to take valid data

# Off-line Flagging and Calibration

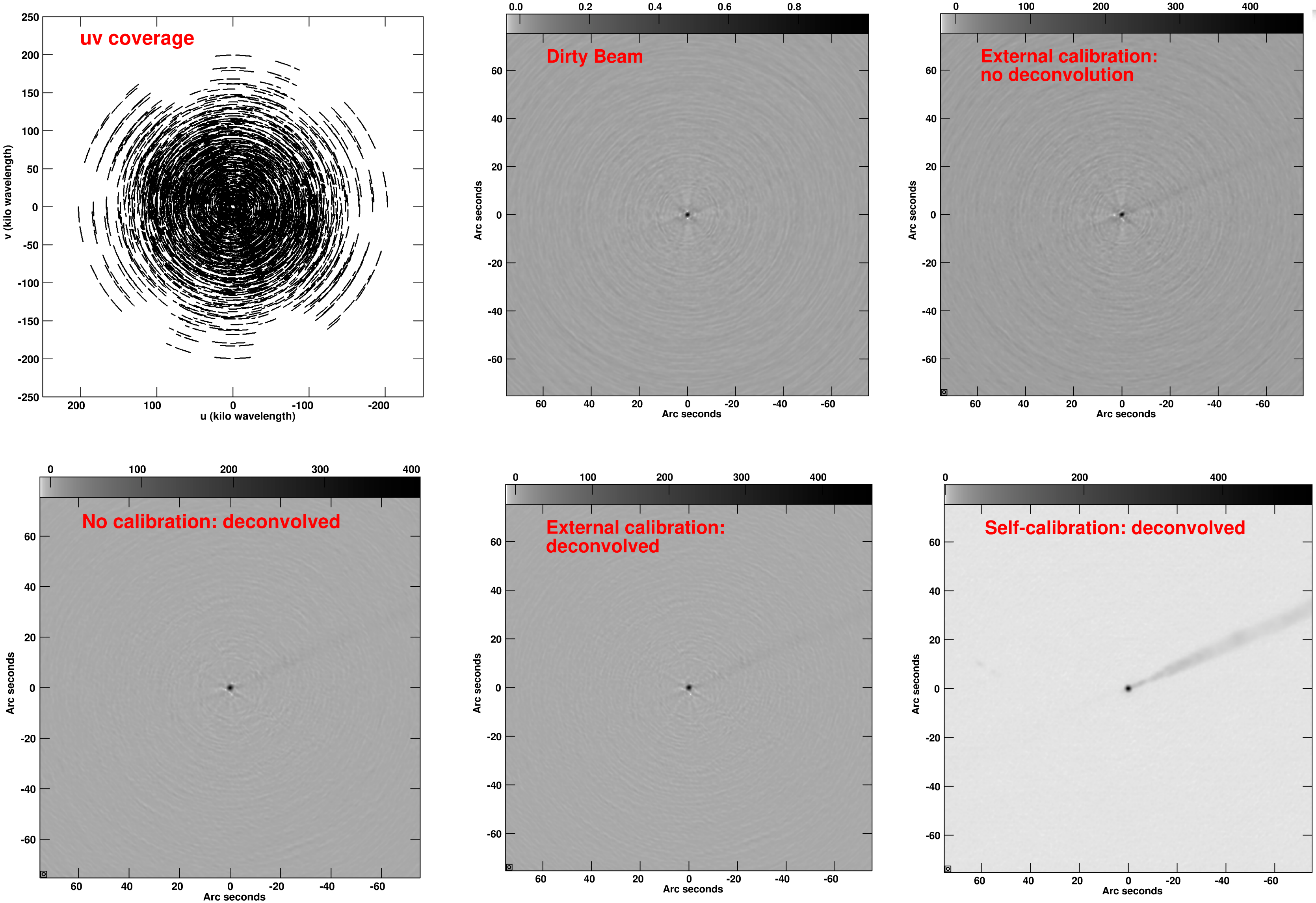
- Residual delay
- Bandpass: variation of complex gain with frequency
- Variation of complex gain with time (instrument, ionosphere, troposphere)
- Flux density scale (standard calibrator)
- Leakage between polarizations

# Imaging




Model Images

CLEAN (represent the sky as a set of delta functions)
Multi-scale clean (represent the sky as a set of Gaussians)
Wavelets
Maximum entropy (maximise a measure of smoothness)


uv coverage
v (kilo wavelength)
u (kilo wavelength)
Dirty Beam
0.0 0.2 0.4 0.6 0.8
External calibration:
no deconvolution
No calibration: deconvolved
External calibration:
deconvolved
Self-calibration: deconvolved
Arc seconds

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