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STELLAR

Scientific and Technological Excellence by Leveraging LOFAR Advancements in Radio Astronomy

ASTRON

Introduction to Radio Interferometry

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DIAS

What is interferometry ?

Measurement

Interference of waves

Where have you seen interference of waves?

On a rainy day

Image credit: Anton/Wikimedia

On a rainy day

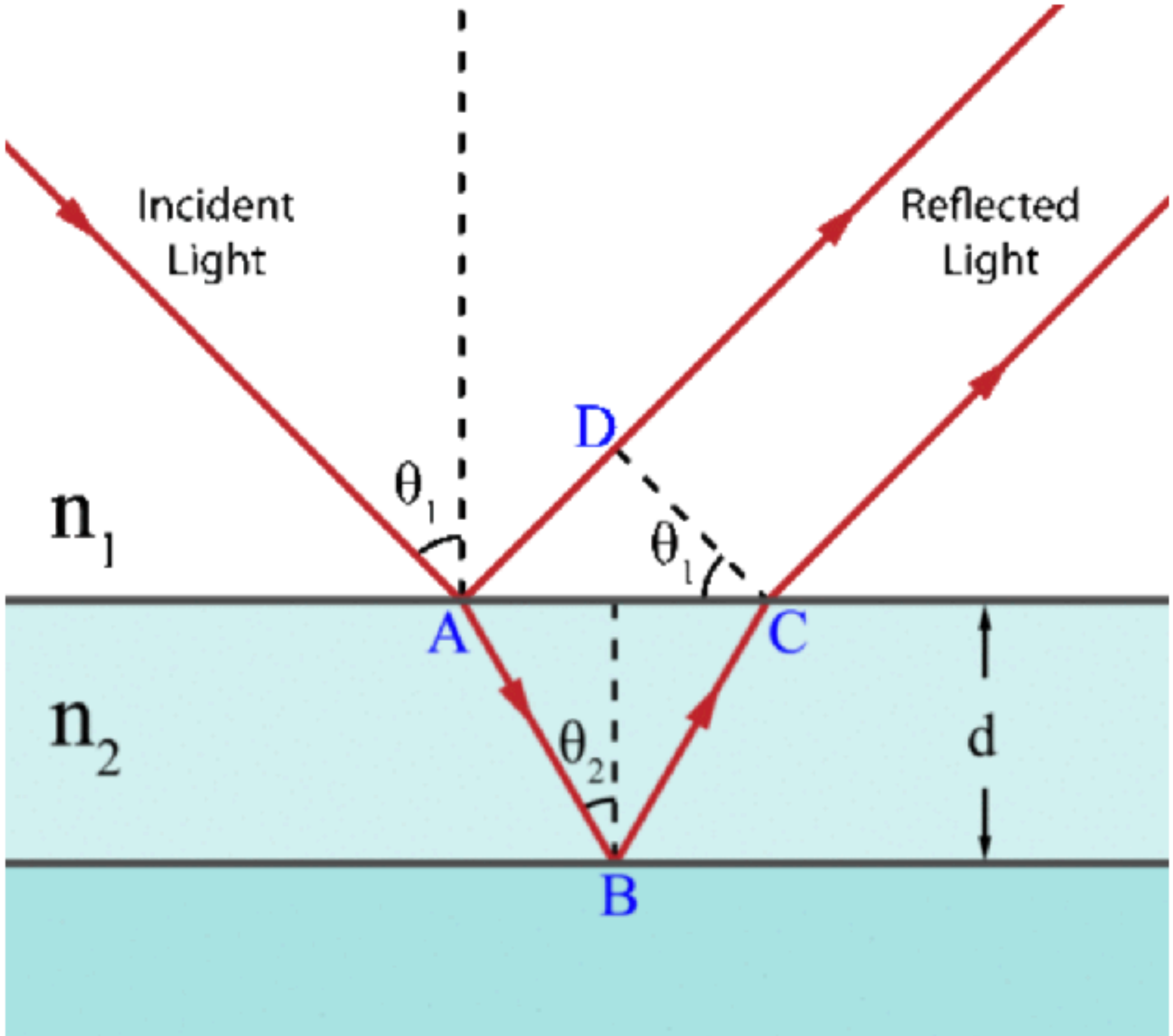


Image credit: Chanli/Wikimedia

Soap bubbles

Image credit: Alvesgaspar/Wikimedia

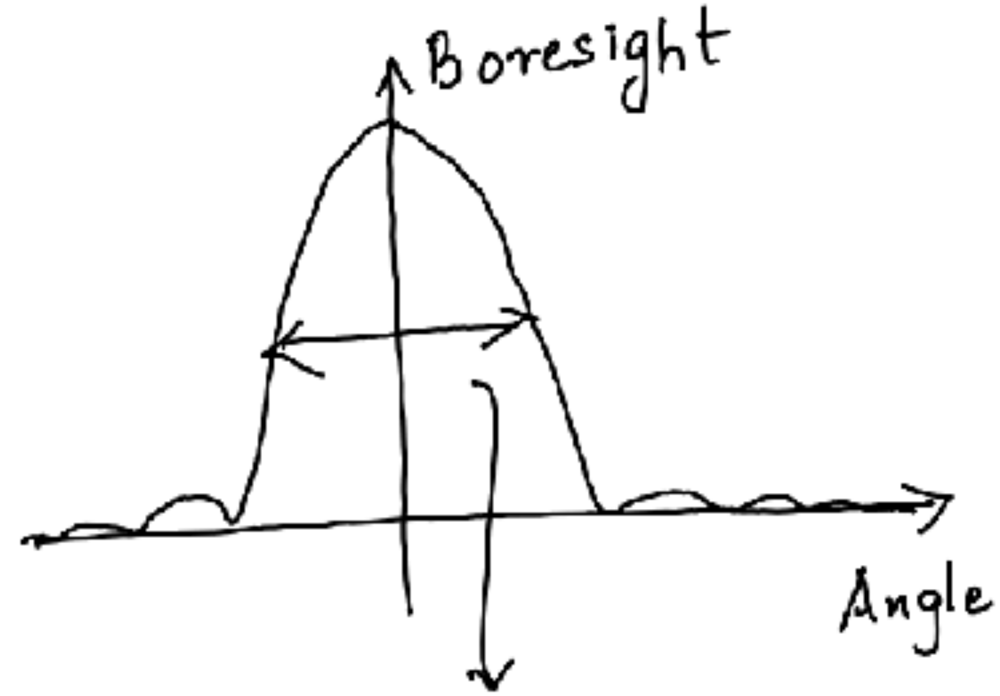
Butterflys, beetles, birds

Image credit: Alexcooper1/Wikimedia

Radio interferometers - why do we build them?

Single dish telescope

Sensitivity on sky



width $\approx \frac{\lambda}{D}$

For LOFAR frequencies;

$\lambda \approx 2\,m$

Say $D \approx 100\,m$

Angular width $= \frac{2}{100} = 0.02\,\text{rad} = 1.15\,\text{deg}$

Radio interferometers - why do we build them?

What if we want 1" resolution?

$$\Delta\theta = 1'' = \frac{1}{3600}\,\text{deg} = \frac{\pi}{180} \times \frac{1}{3600}\,\text{radian} = 4.85 \times 10^{-6}$$

$$\Delta\theta = \frac{\lambda}{D} \Rightarrow D = \frac{\lambda}{\Delta\theta}$$

$\lambda = 2$ metres

$$D = \frac{2}{4.85 \times 10^{-6}} = 4.12 \times 10^{5}\ \text{metre}$$

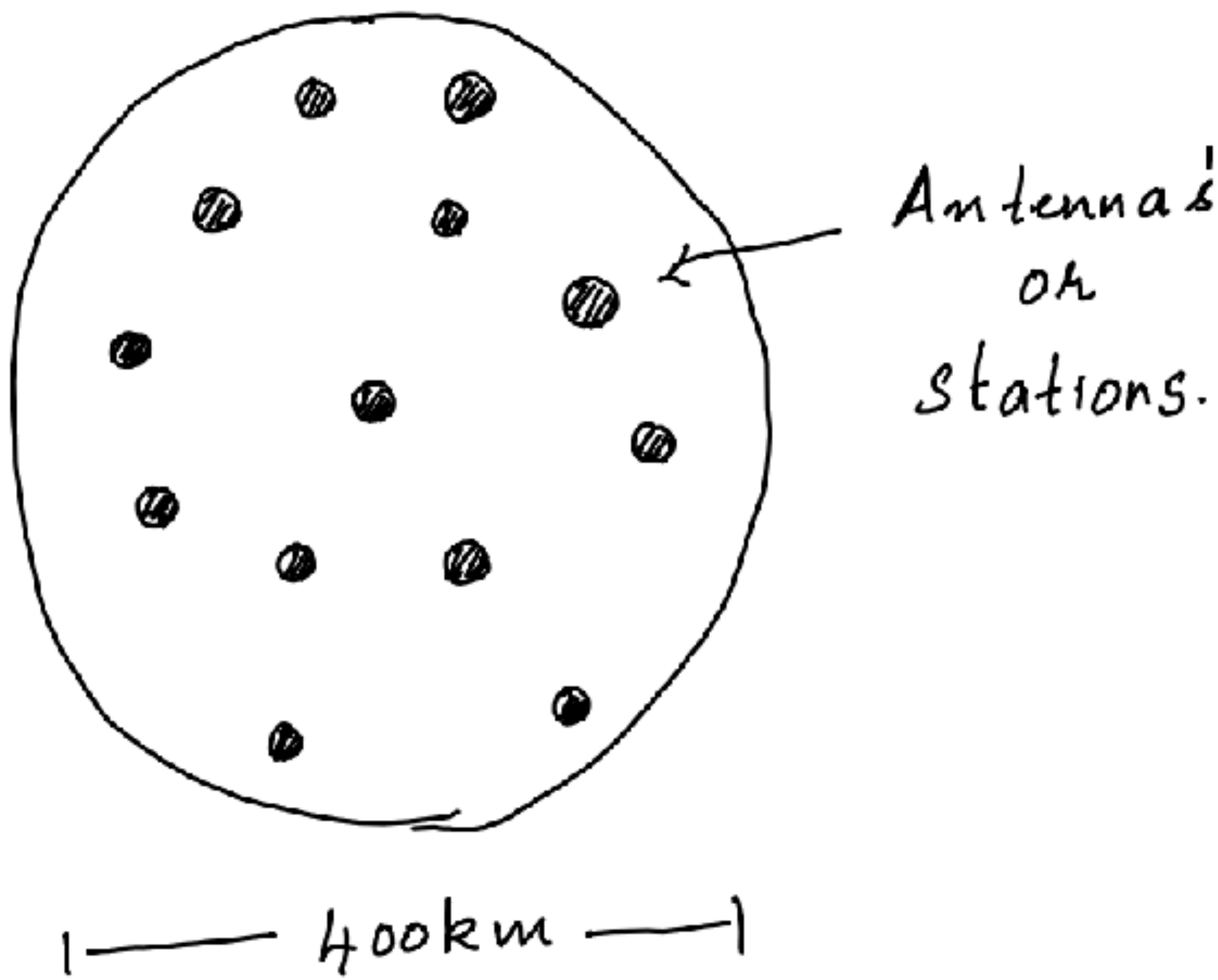
$= 412$ km !!!

We cannot build such a dish!

Radio interferometers - small part of a large aperture

Solution :-

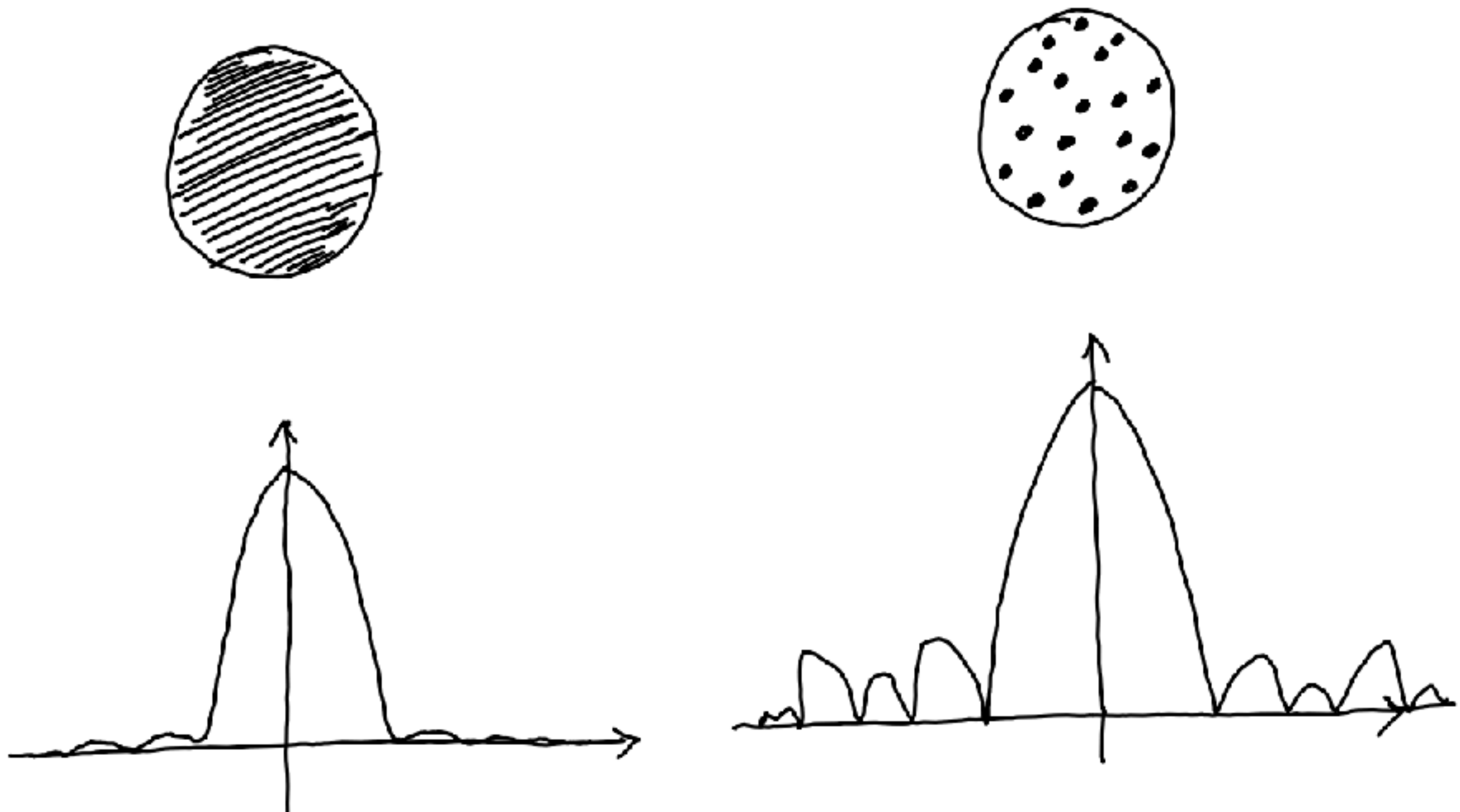
We can buid a small part of the dish!



Radio interferometers - the tradeoff

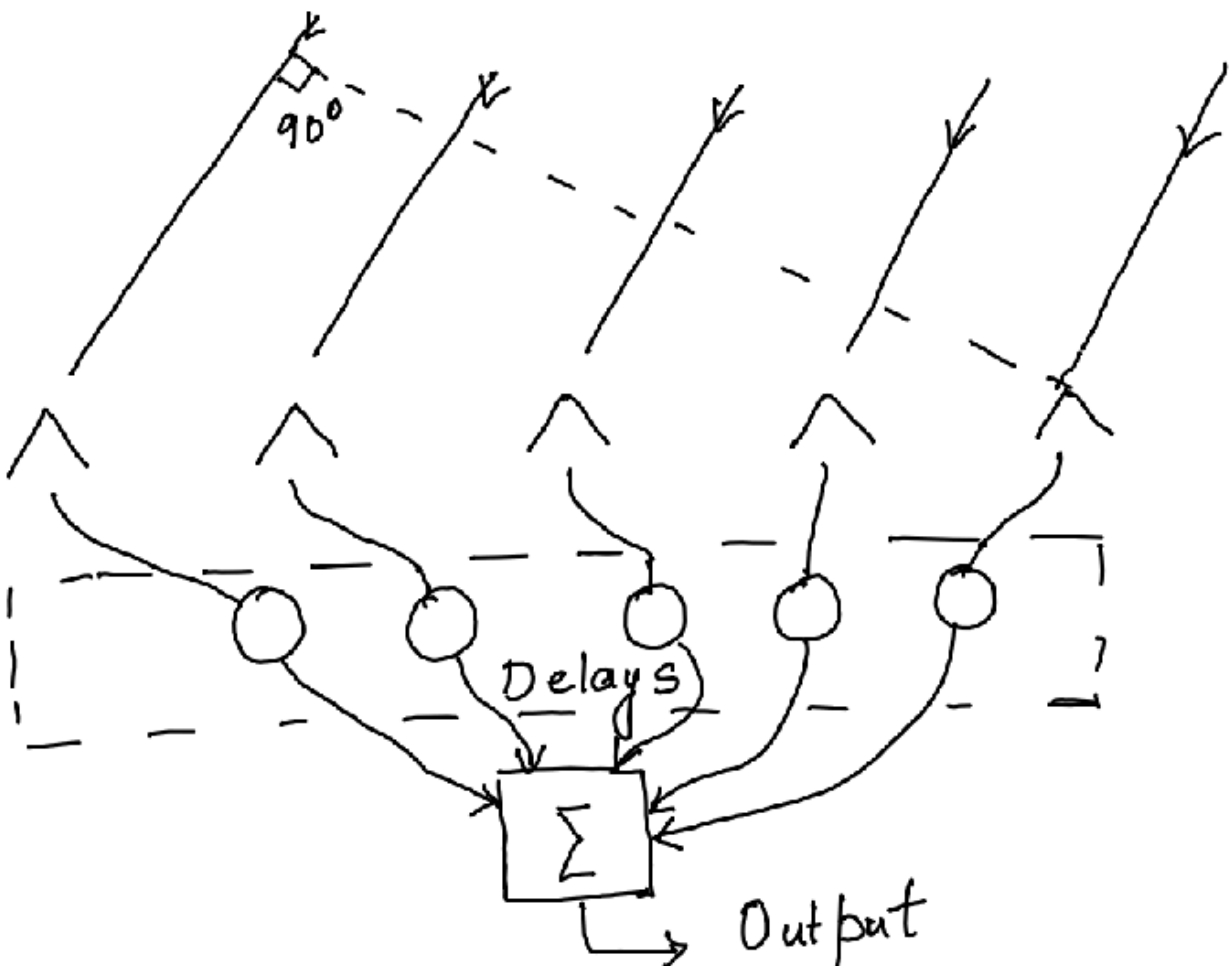
Same angular resolution

But worse beam

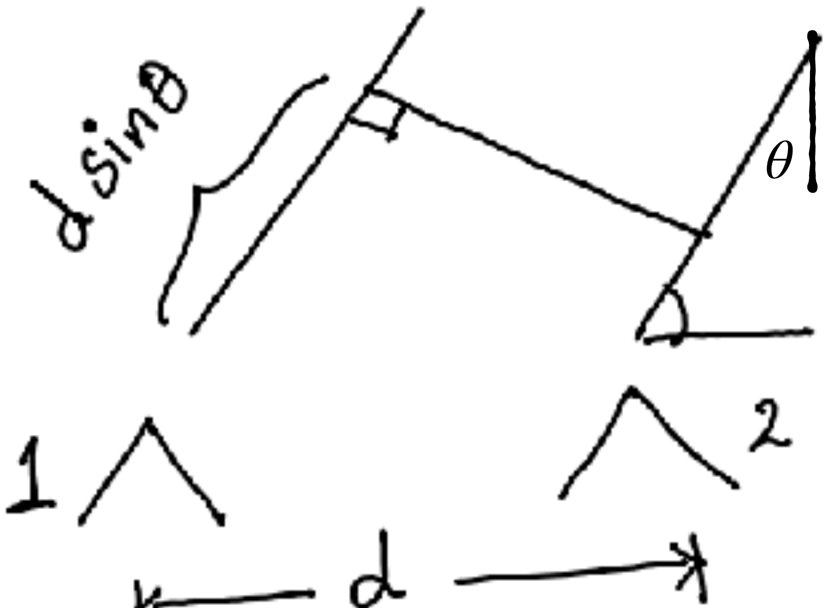


How to combine the signals of the antennas?

Option-1 : Beamforming



Mathematics of Beamforming



Extra path length $= d\sin\theta$

Electric field at A1: $\mathcal{E}_0 e^{j\omega t} e^{-j\omega d\sin\theta}$

A2: $\mathcal{E}_0 e^{j\omega t}$

Delay line at A1 : Output delayed by 0

A2 : Output delayed by $t = d\sin\theta_2$

Final output power $= \left| \mathcal{E}_0 e^{j\omega t} e^{-j\omega d\sin\theta} + \mathcal{E}_0 e^{j\omega t} e^{-j\omega d\sin\theta} \right|$

$= 4\mathcal{E}_0^2$

Generalizing to N elements

$\hat{A1} \; \_d\_ \; \hat{A2} \; \_d\_ \; \hat{A3} \; \ldots\ldots \; \hat{A_N}$

$$\text{Output} = \left| \varepsilon_1 + \varepsilon_2 e^{-j\omega d\sin\theta} + \varepsilon_3 e^{-j2\omega d\sin\theta} + \ldots\ldots + \varepsilon_N e^{-j(N-1)d\omega\sin\theta} \right|^2$$

$$= \left| \sum_{i=1}^{N} \varepsilon_i \, e^{-j(i-1)\omega d\sin\theta} \right|^2$$

$$= \left[\sum_{i=1}^{N} \varepsilon_i e^{-j(i-1)\omega d\sin\theta} \right] \underbrace{\left[\sum_{i=1}^{N} \varepsilon_i \, e^{-j(i-1)\omega d\sin\theta} \right]^*}_{\text{Complex Conjugate}}$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{N} \underbrace{\mathcal{E}_i \mathcal{E}_k^*}_{} \, e^{\overbrace{-j(i-k)\,\omega d \sin\theta}^{}}$$

$\mathcal{E}_i \mathcal{E}_k^*$ → 2-point correlation of the incident electric field

↓

Intrinsic property of the incident sky signal

$e^{-j(i-k)\omega d \sin\theta}$ → Complex phasor that only depends on viewing geometry (θ) and array geometry (d)

<u>Option-2</u> :- → Record $\mathcal{E}_i \mathcal{E}_k^*$ for all pairs (i,k)

→ "Beamform" later in software

$$\sum_{i=1}^{N} \sum_{k=1}^{N} \mathcal{E}_i \mathcal{E}_k^* \; e^{-j(i-k)\omega d \sin\theta}$$

Any choice of (i,k) is called a <u>Baseline</u>

$\mathcal{E}_i \mathcal{E}_k^*$ is called the <u>visibility</u>, V_{ik}

$(i-k)d$ is the baseline length

$(i-k)d\sin\theta$ is the projected baseline length

In astronomy, the sources are at large distances, so $\mathcal{E}_i \mathcal{E}_k^*$ must only depend on $(i-k)$ ie the baseline vector and not your exact position on the Earth

Notice also that

$$\sum_i \sum_k V(i-k)\, e^{-j(i-k)\omega d \sin\theta}$$

closely resembles a Fourier transform.

We will now formally derive the fundamental equation of radio interferometry for an arbitrary 2D array of antennas

Notice also that

$$\sum_i \sum_k V(i-k)\, e^{-j(i-k)\omega d \sin\theta}$$

closely resembles a Fourier transform.

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Formal derivation of the basic equation of interferometry

Let $I(\vec{s})$ be the brightness of the sky in direction $\vec{s}$

{ I has units of $\frac{erg}{s\ Hz\ cm^2\ sr}$ }

i.e. $I = \frac{dE}{dA\, dt\, d\nu\, d\Omega}$

- dE ← Energy
- dA ← Area on Earth
- dt ← time
- $d\nu$ ← Bandwidth
- $d\Omega$ ← Solid angle

$I(\vec{s})$ is related to the electric field E by Poynting's theorem

$$I(\vec{s}) = \frac{c}{4\pi} \left| \vec{E}(\vec{s}) \right|^2$$

Because astronomical sources are so far their EM radiation reaches us as plane waves.

$$\vec{E}(\vec{s}, \vec{R}) = \sqrt{\frac{I(\vec{s})\, 4\pi}{c}}\; e^{j\omega t}\; e^{-j\omega \left[\frac{\vec{R}\cdot\vec{s}}{c}\right]}$$

← Extra geometric path delay

The Electric field due to all directions is

$$\vec{E}(\vec{R}) = \int d\Omega \sqrt{\frac{I(\vec{s})\, 4\pi}{c}}\; e^{j\omega t}\; e^{-j\omega \frac{\vec{R}\cdot\vec{s}}{c}}$$

The visibility is defined as

$$V(\vec{R}_1, \vec{R}_2) = \frac{c}{4\pi}\left\langle E(R_1)\, E^*(R_2)\right\rangle \leftarrow \text{Ensemble average (Mean value)}$$

$$V(\vec{R}_1, \vec{R}_2) = \int d\Omega_1 \int d\Omega_2 \left\langle \sqrt{I(\vec{s}_1)\, I(\vec{s}_2)} \right\rangle e^{-\frac{j\omega}{c}\left[\vec{R}_1 \cdot \vec{s}_1 - \vec{R}_2 \cdot \vec{s}_2\right]}$$

Because one location on the sky emits independently of the other

$I(\vec{s}_1)$ and $I(\vec{s}_2)$ are uncorrelated

$$V(\vec{R}_1, \vec{R}_2) = \int d\Omega_1 \int d\Omega_2\, I(\vec{s}_1)\, \delta(\vec{s}_1 - \vec{s}_2)\, e^{-j\frac{\omega}{c}\left[\vec{R}_1 \cdot \vec{s}_1 - \vec{R}_2 \cdot \vec{s}_2\right]}$$

$$V(\vec{R}_1, \vec{R}_2) = \int d\Omega\, I(\vec{s})\, e^{-\frac{j\omega}{c}(\vec{R}_1 - \vec{R}_2)\cdot\vec{s}}$$

which only depends on $\vec{R}_1 - \vec{R}_2$ as we anticipated

$$V(\vec{R}_1 - \vec{R}_2) = \int d\Omega\, I(\vec{s})\, e^{-j\frac{\omega}{c}\vec{s}\cdot(\vec{R}_1 - \vec{R}_2)}$$

Let $\vec{B} = \vec{R}_1 - \vec{R}_2$ be the baseline vector

$$V(\vec{B}) = \int d\Omega\, I(\vec{s})\, e^{-j\frac{\omega \vec{s}}{c}\cdot\vec{B}}$$

This shows that the visibilities are the Fourier transform of the sky brightness distribution.

$\vec{s}$ is a unit vector sky position.

Its Cartesian Components are

$l = \vec{s}\cdot\hat{x}$ (x component)

$m = \vec{s}\cdot\hat{y}$ (Y Component) and $n = \sqrt{1 - l^2 - m^2}$ (z Component)

Similarly let all the baselines lie on a 2D plane on Earth

$$\frac{\vec{B}\cdot\hat{x}}{\lambda} = u \quad (x \text{ Component})$$

$$\frac{\vec{B}\cdot\hat{y}}{\lambda} = v \quad (Y \text{ Component})$$

The elemental solid angle is $d\Omega = \sin\theta \, d\theta \, d\phi$.

where $l = \sin\theta\cos\phi$

$m = \sin\theta\sin\phi$

$n = \cos\theta$

$$dl\,dm = \begin{vmatrix} \frac{\partial l}{\partial\theta} & \frac{\partial l}{\partial\phi} \\ \frac{\partial m}{\partial\theta} & \frac{\partial m}{\partial\phi} \end{vmatrix} d\theta\, d\phi$$

$$\frac{\partial l}{\partial \theta} = \cos\theta \cos\phi \qquad \frac{\partial l}{\partial \phi} = -\sin\theta \sin\phi$$

$$\frac{\partial m}{\partial \theta} = \cos\theta \sin\phi \qquad \frac{\partial m}{\partial \phi} = \sin\theta \cos\phi$$

$$\begin{vmatrix} \frac{\partial l}{\partial \theta} & \frac{\partial l}{\partial \phi} \\ \frac{\partial m}{\partial \theta} & \frac{\partial m}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta \cos\phi & -\sin\theta \sin\phi \\ \cos\theta \sin\phi & \sin\theta \cos\phi \end{vmatrix}$$

$$= \sin\theta \cos\theta \cos^2\phi + \sin\theta \cos\theta \sin^2\phi$$

$$= 2 \sin\theta \cos\theta$$

$$\therefore \quad dl\,dm = 2 \sin\theta \cos\theta \, d\theta \, d\phi = 2 \cos\theta \, d\Omega$$

$$d\Omega = \frac{dl\,dm}{2\cos\theta} = \frac{dl\,dm}{2\sqrt{1-l^2-m^2}}$$

The factor of 2 is dropped because we only observe one hemisphere

Basic equation of interferometry

$$V(u,v) = \int \mathrm{d}l \int \mathrm{d}m \frac{I(l,m)}{\sqrt{1-l^2-m^2}} \exp\left[-j(ul+vm)\right]$$

… a 2D Fourier transform

The inverse transform is ..

$$\frac{I(l,m)}{\sqrt{1-l^2-m^2}} = \int \mathrm{d}u \int \mathrm{d}v V(u,v) \exp\left[j(ul+vm)\right]$$

Primary antenna beam

So far we assumed that the interferometer elements are isotropic receivers….

If they have a beam given by $A(l, m)$ then …

$$V(u, v) = \int \mathrm{d}l \int \mathrm{d}m \frac{I(l, m)A(l, m)}{\sqrt{1 - l^2 - m^2}} \exp\left[-j(ul + vm)\right]$$

$$\frac{I(l, m)A(l, m)}{\sqrt{1 - l^2 - m^2}} = \int \mathrm{d}u \int \mathrm{d}v V(u, v) \exp\left[j(ul + vm)\right]$$

$A(l, m)$ is called the "primary beam"

Earth rotation synthesis

So far we have neglected Earth's rotation

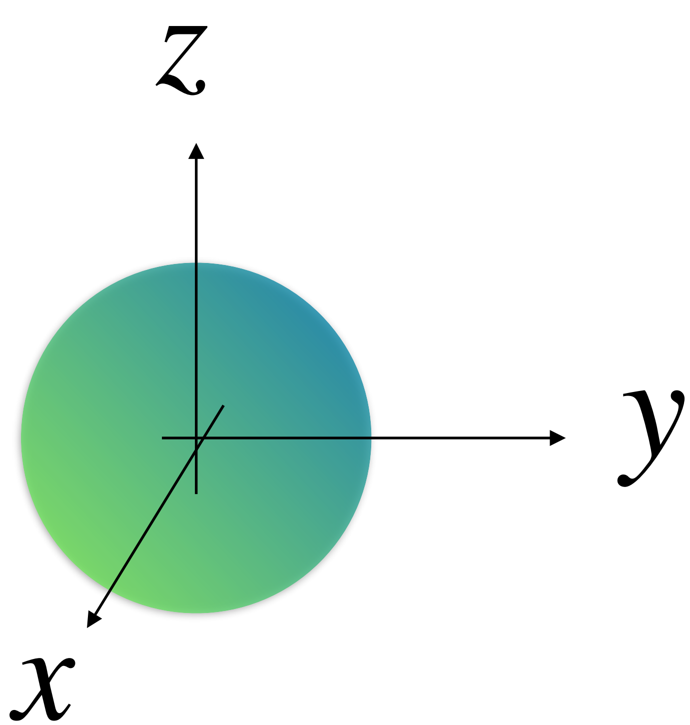
Topocentric coordinates

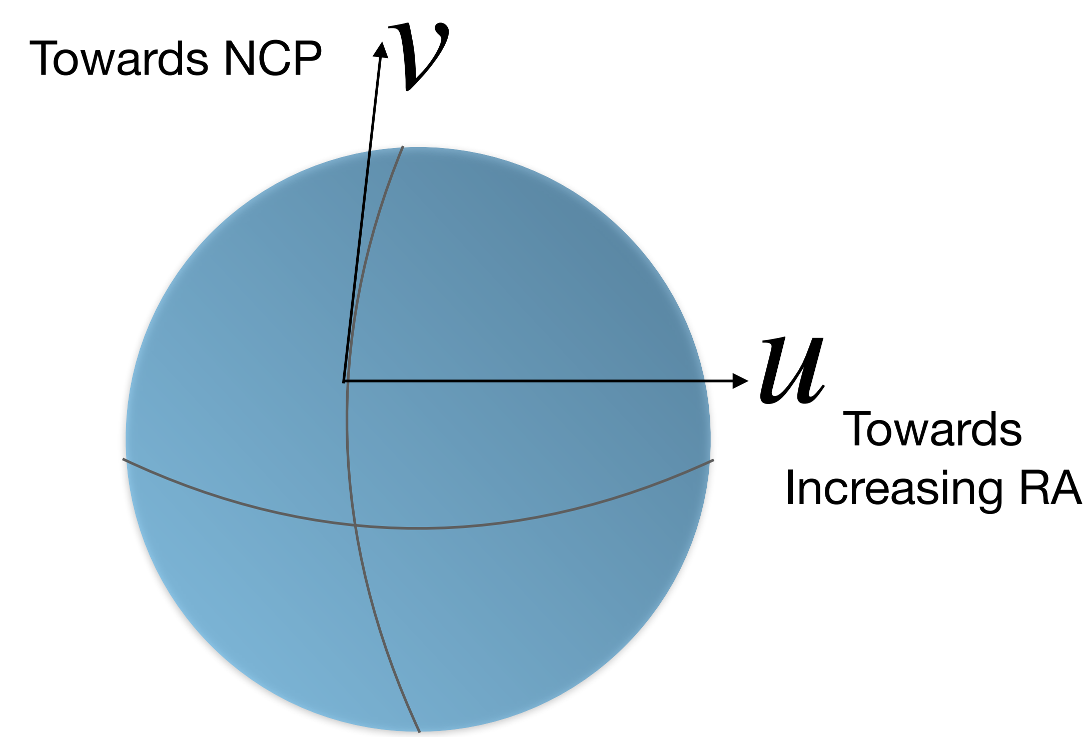
Geometrical transform

$f(x, y, z, t, \mathrm{RA}, \mathrm{DEC}) = (u, v)$

→

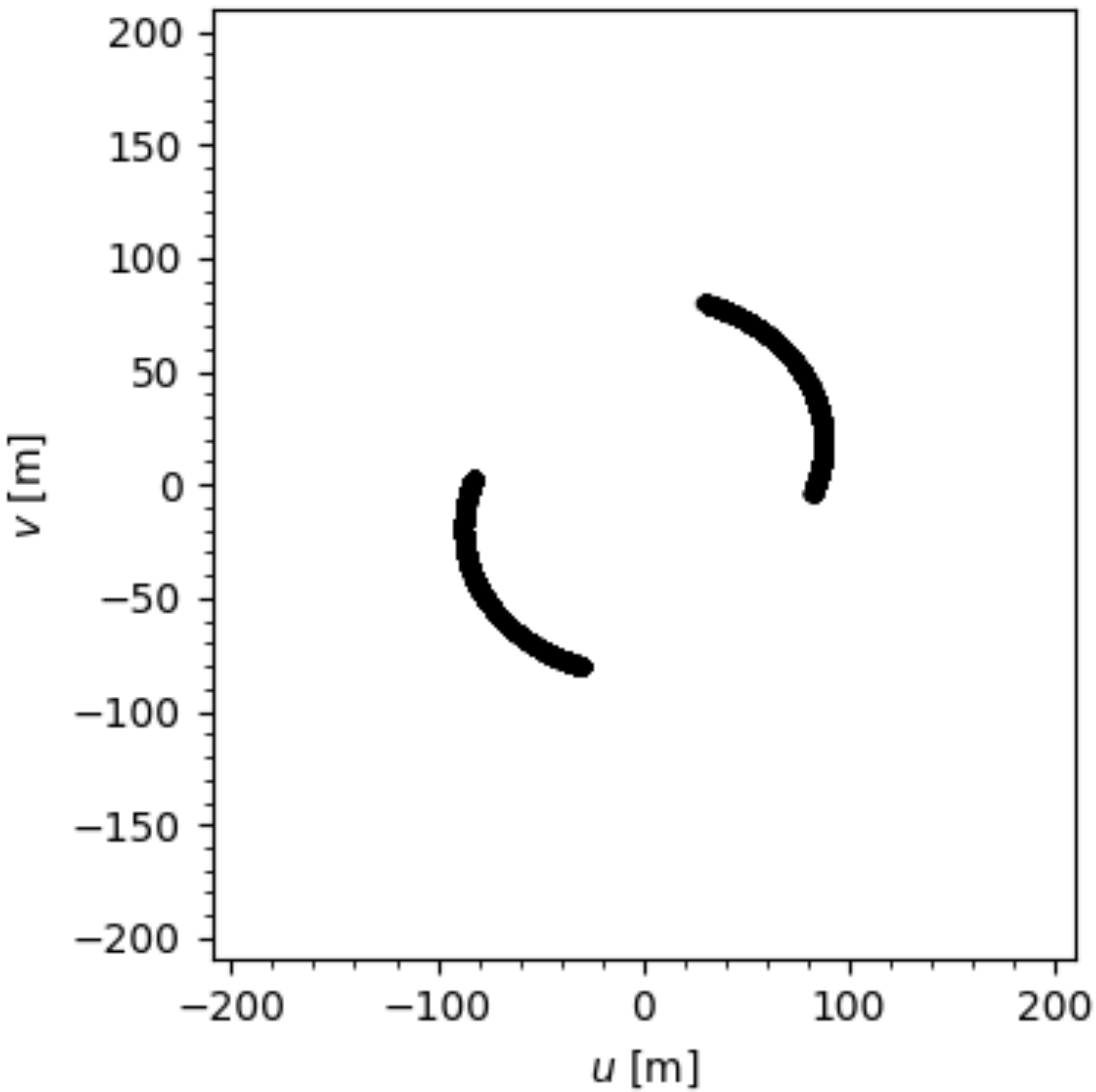
Celestial coordinates





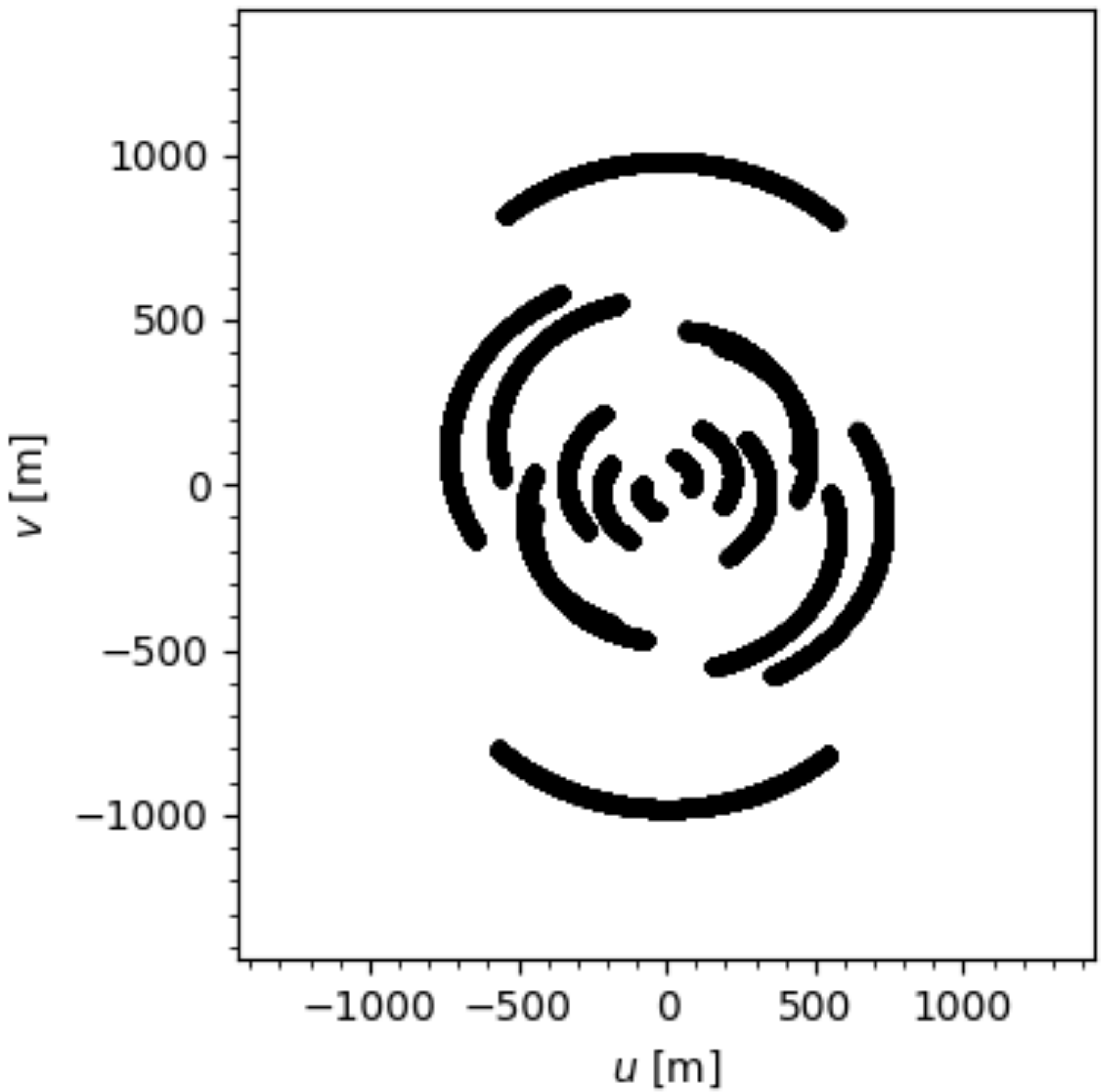
Earth rotation synthesis

Baselines traverse ellipses in the (u,v) plane



Earth rotation synthesis

Several baselines and Earth rotation fill up the uv-plane

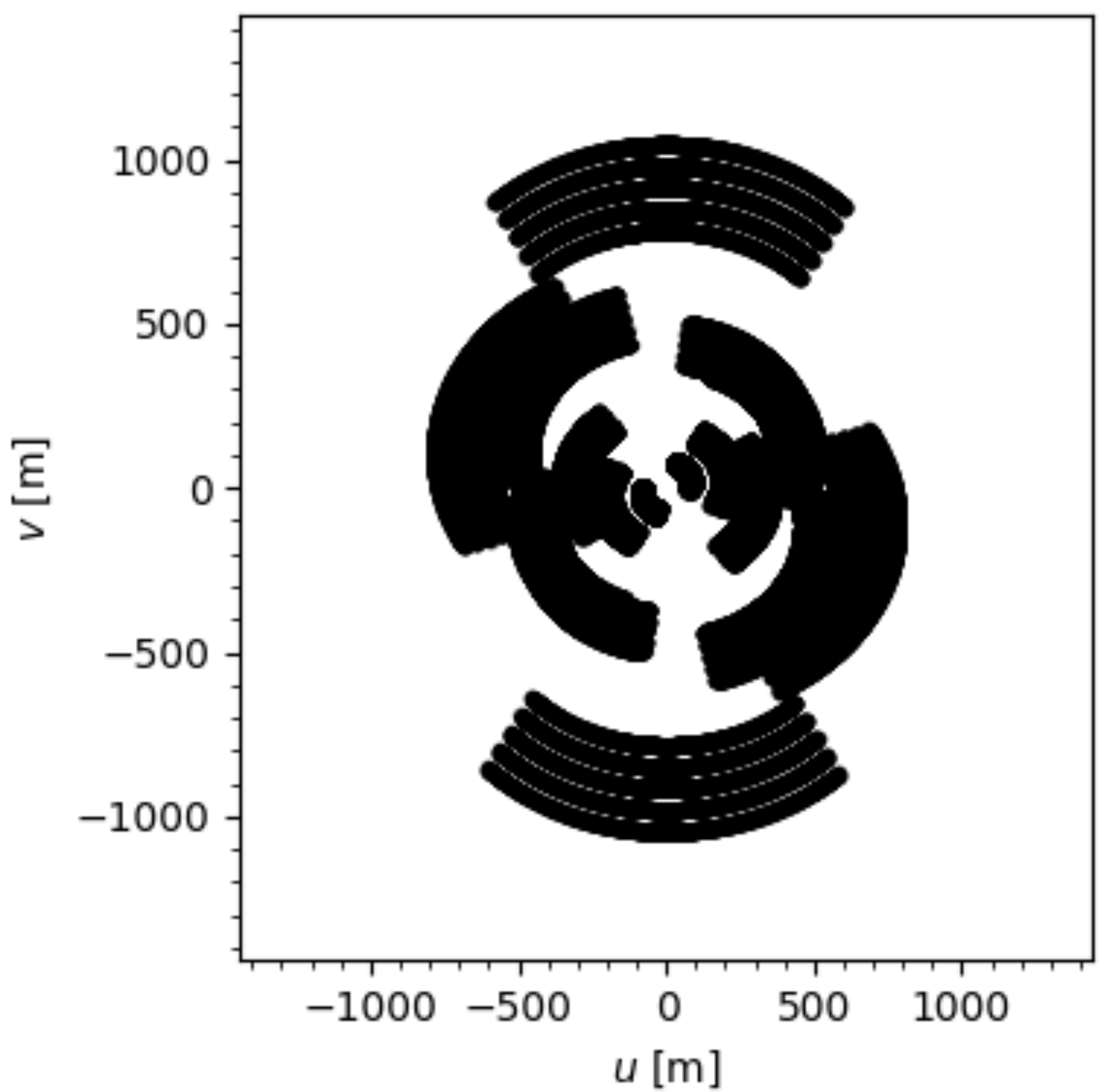


Bandwidth synthesis

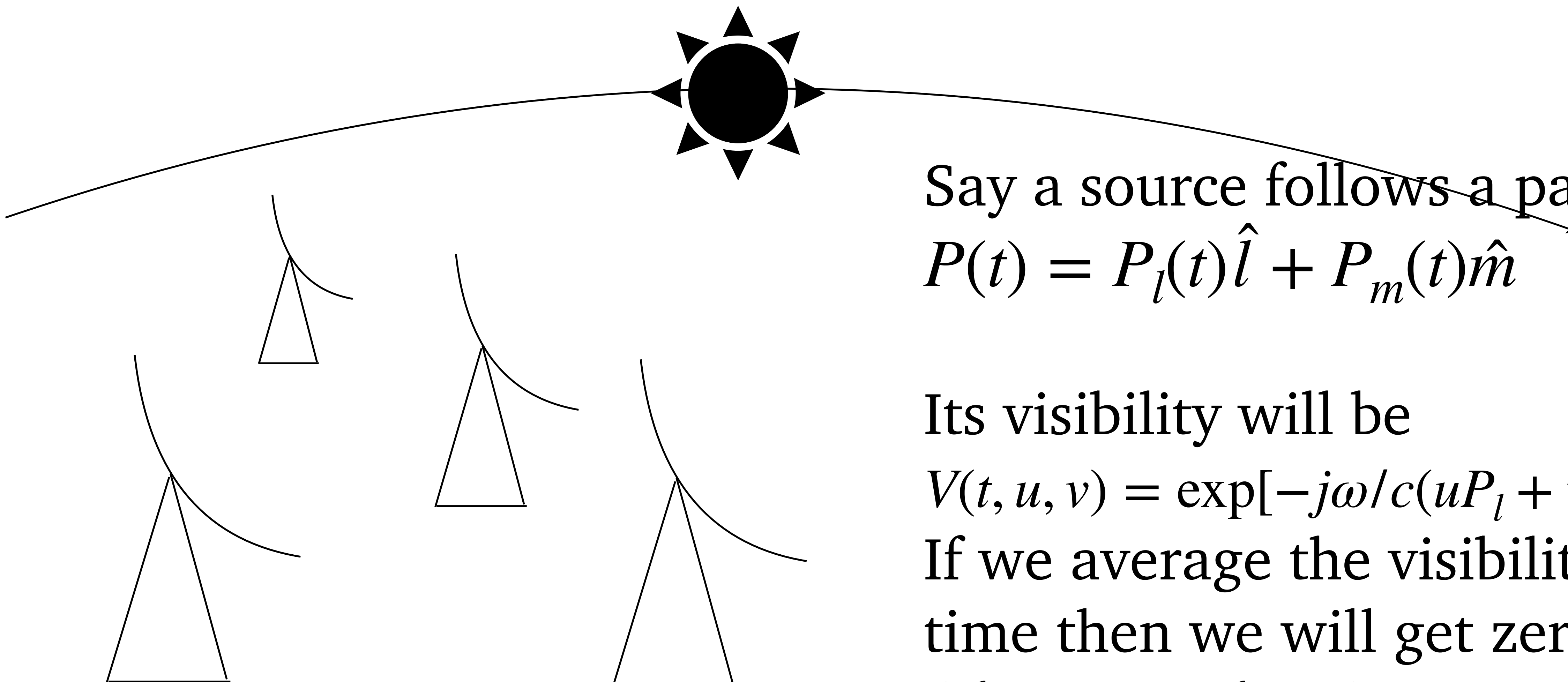
So far we have assumed mono-chromatic waves

$$u = x/\lambda = x\nu/c; \quad v = y\nu/c$$

We can do Earth rotation + bandwidth synthesis



Time smearing

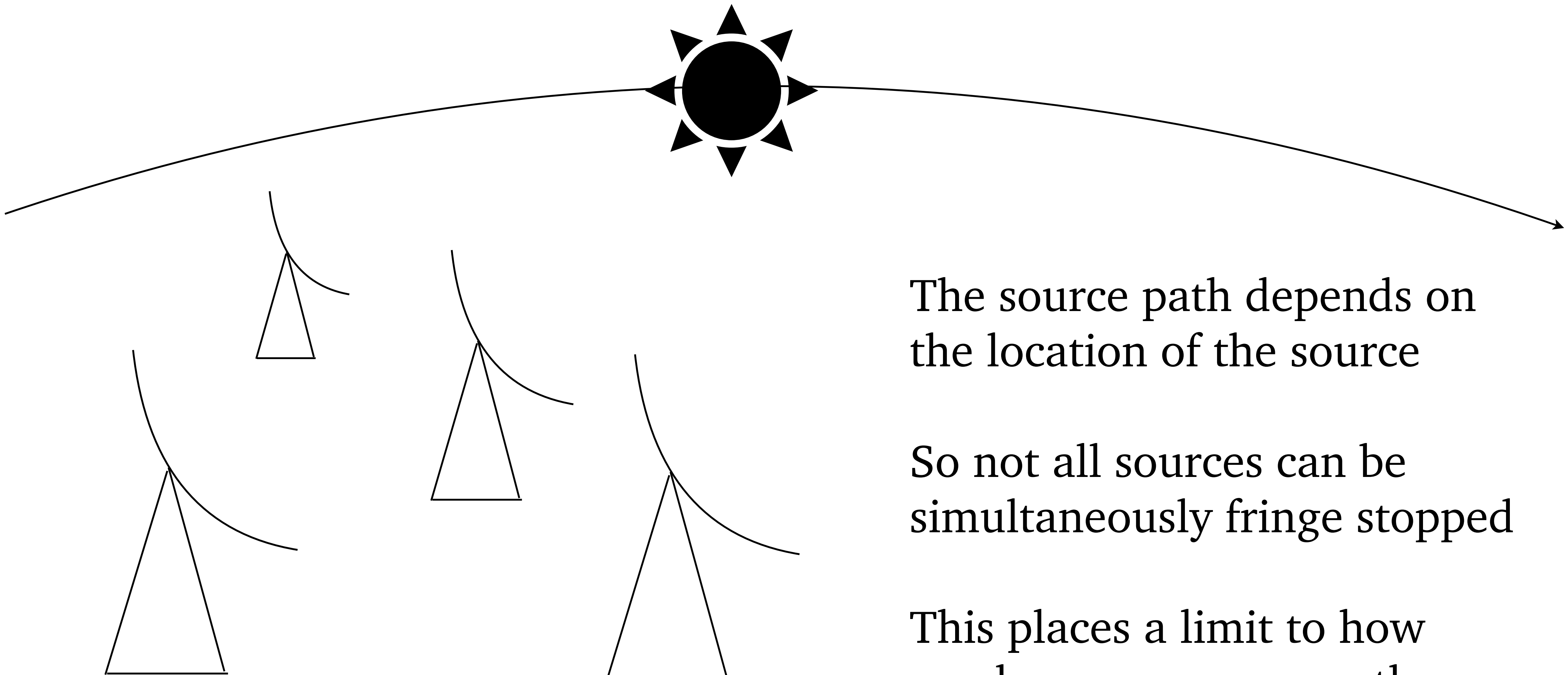


Say a source follows a path
$P(t) = P_l(t)\hat{l} + P_m(t)\hat{m}$

Its visibility will be
$V(t, u, v) = \exp[-j\omega/c(uP_l + vP_m)]$
If we average the visibility in time then we will get zero (changing phase)!

To avoid this, we apply the phase $\exp[j\omega/c(P_l u + P_m v)]$ and then average.
This is called Fringe stopping.

Time smearing



The source path depends on the location of the source

So not all sources can be simultaneously fringe stopped

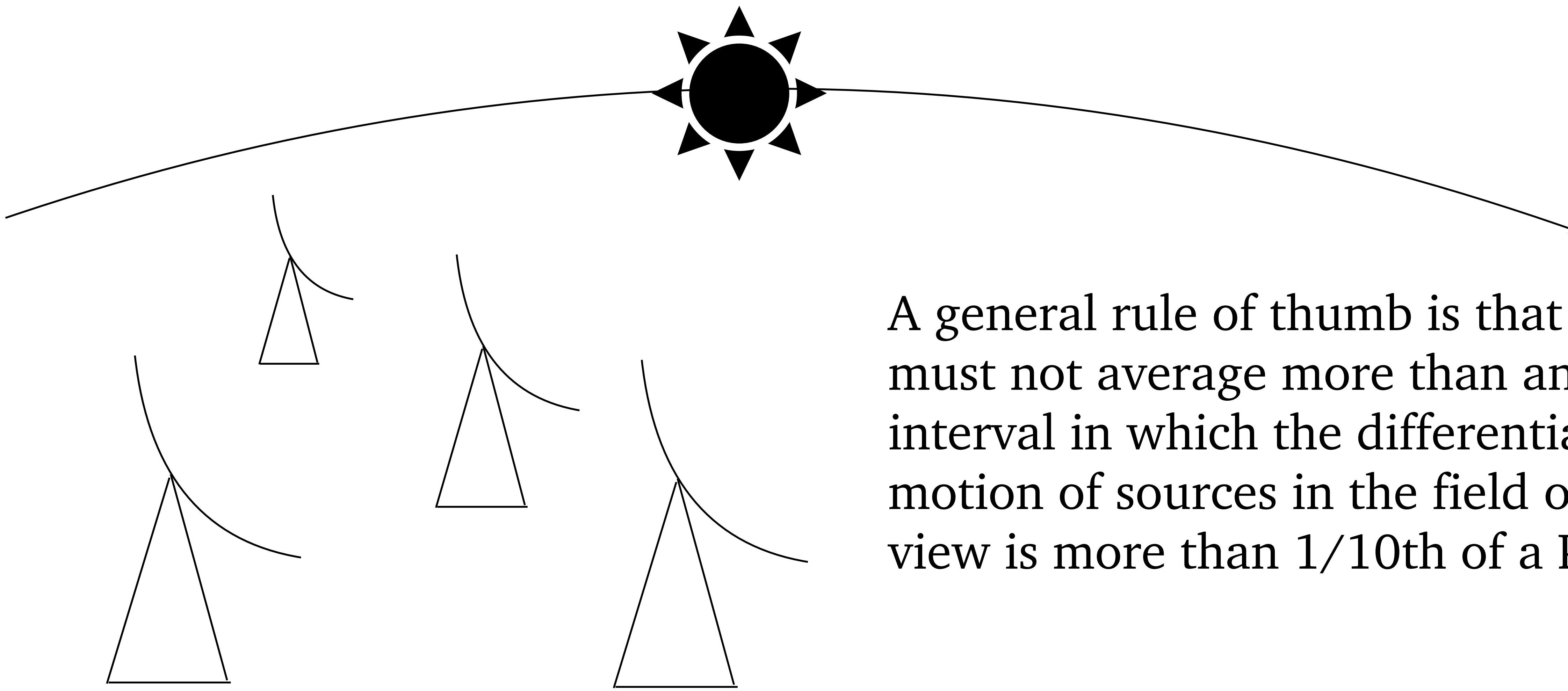
This places a limit to how much one can average the visibility.

Say you image at a declination of δ with a field of view of $\Delta\delta$

The centre of the field moves across the sky at a speed of $\cos\delta \times 15''$ per second

Sources at the top and bottom edges move at a speed of $\cos(\delta \pm \Delta\delta) \times 15''$ per second

Time smearing - rule of thumb



A general rule of thumb is that one must not average more than an interval in which the differential motion of sources in the field of view is more than 1/10th of a PSF

Say max baseline = $10^3\lambda$; 1/10 * PSF width $\approx 20''$

Say $\delta = 45°$ and $\Delta\delta = 5°$ then the differential rate of motion is $\approx 1''$ per second

So we cannot average for more than 20 seconds to avoid time smearing.

Bandwidth smearing

Say you want to properly image a source that is Δl off the fringe-stop centre

Its phase angle on a physical baseline will be $\dfrac{2\pi x}{c}\Delta l \Delta \nu$

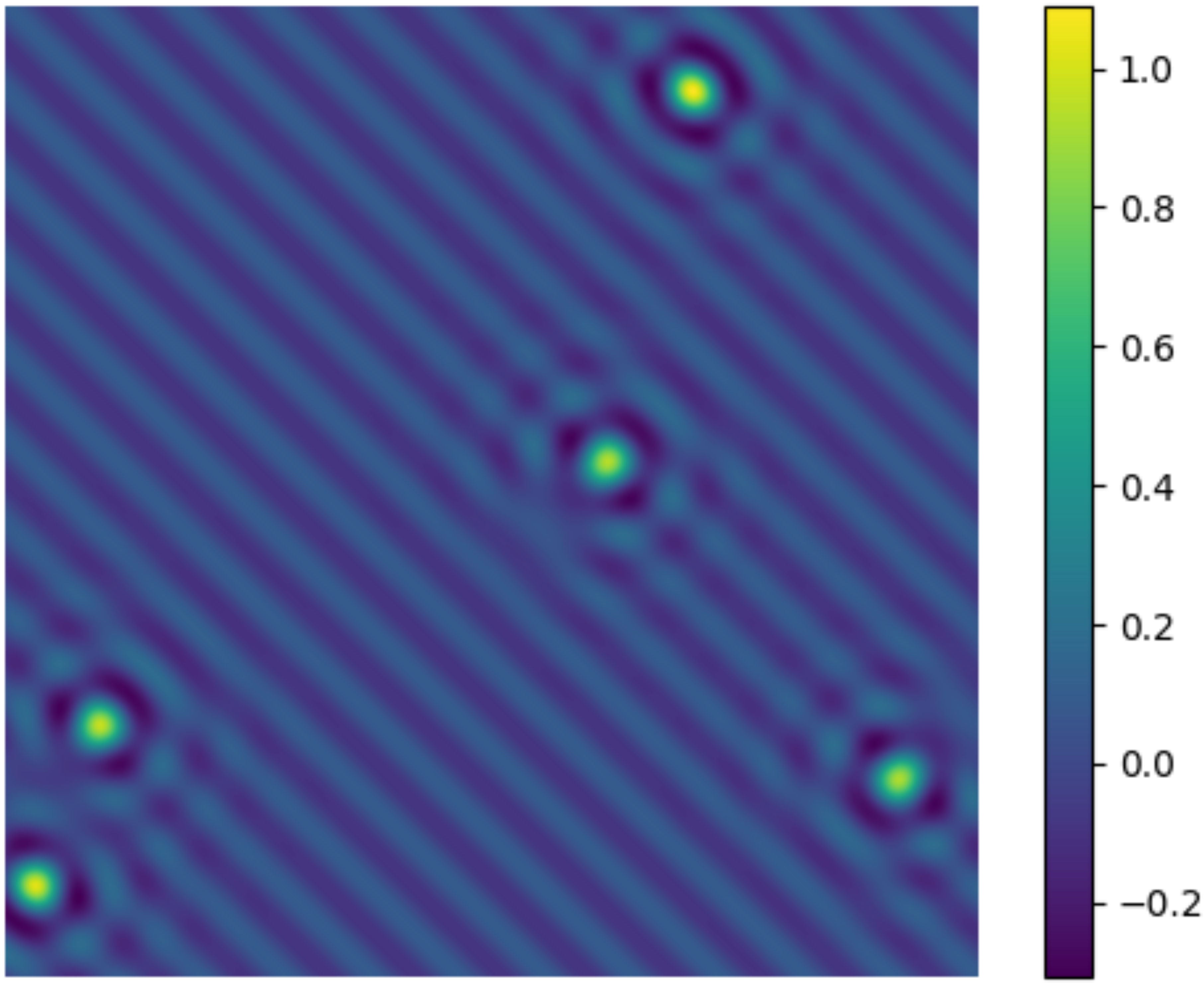
If we average the visibility over a channel of width $\Delta \nu = \dfrac{c}{2\pi x \Delta l}$ then we will average the flux of the source to 0 (source becomes invisible!)

A rule of thumb is to choose $\Delta \nu < \dfrac{1}{10}\dfrac{c}{2\pi x \Delta l}$

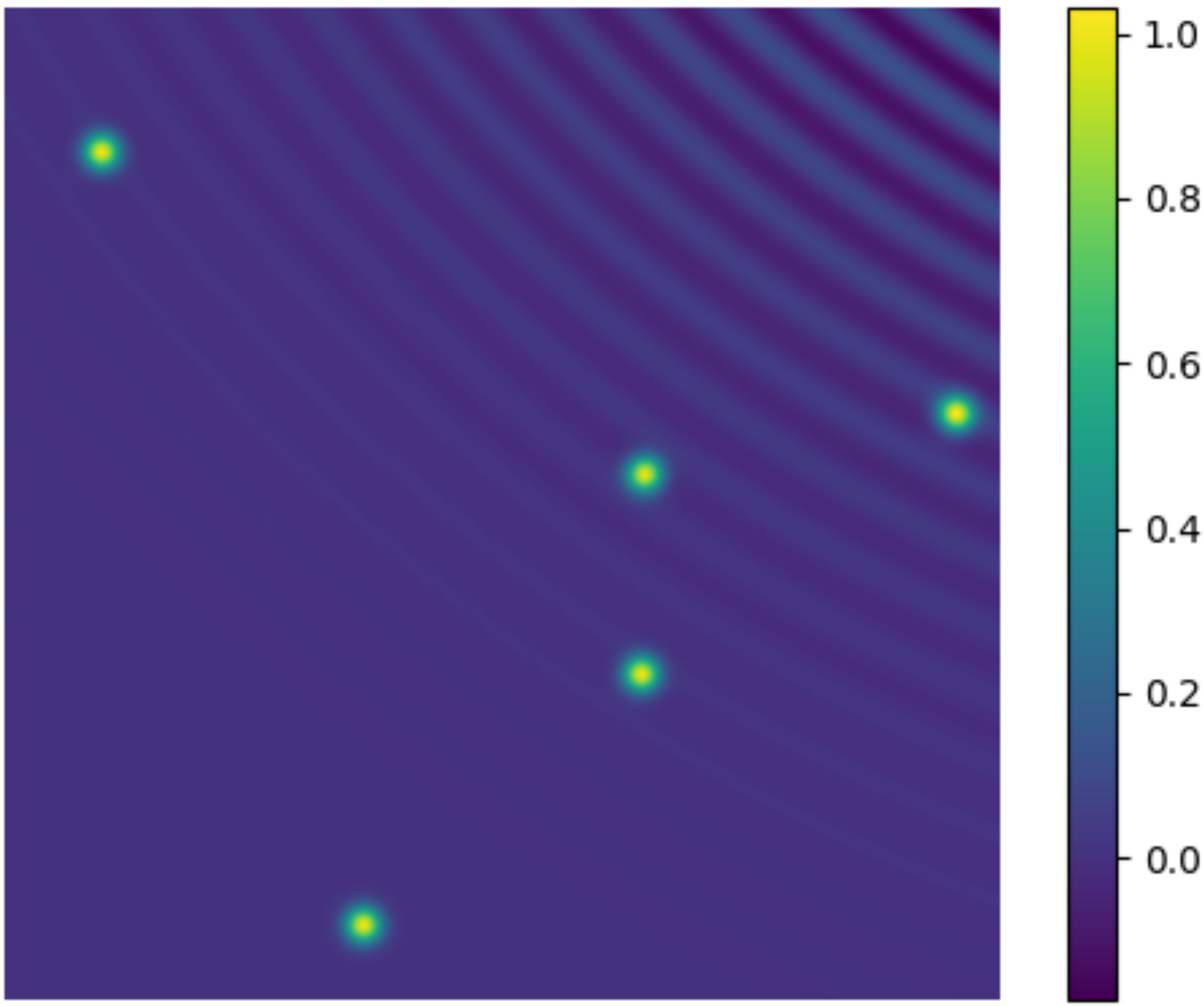
We can take Δl to be $\dfrac{c}{\nu D}$ where D is the station diameter so we get to image the whole field of view properly without flux loss.

For LOFAR $D = 30\,\mathrm{m}$, $\nu = 150\,\mathrm{MHz}$, which gives $\Delta \nu < 10\,\mathrm{kHz}\left(\dfrac{x}{10\,\mathrm{km}}\right)^{-1}$

Thinking in Fourier space - what went wrong?



Thinking in Fourier space - what went wrong?



Appendix: Some basic calculations

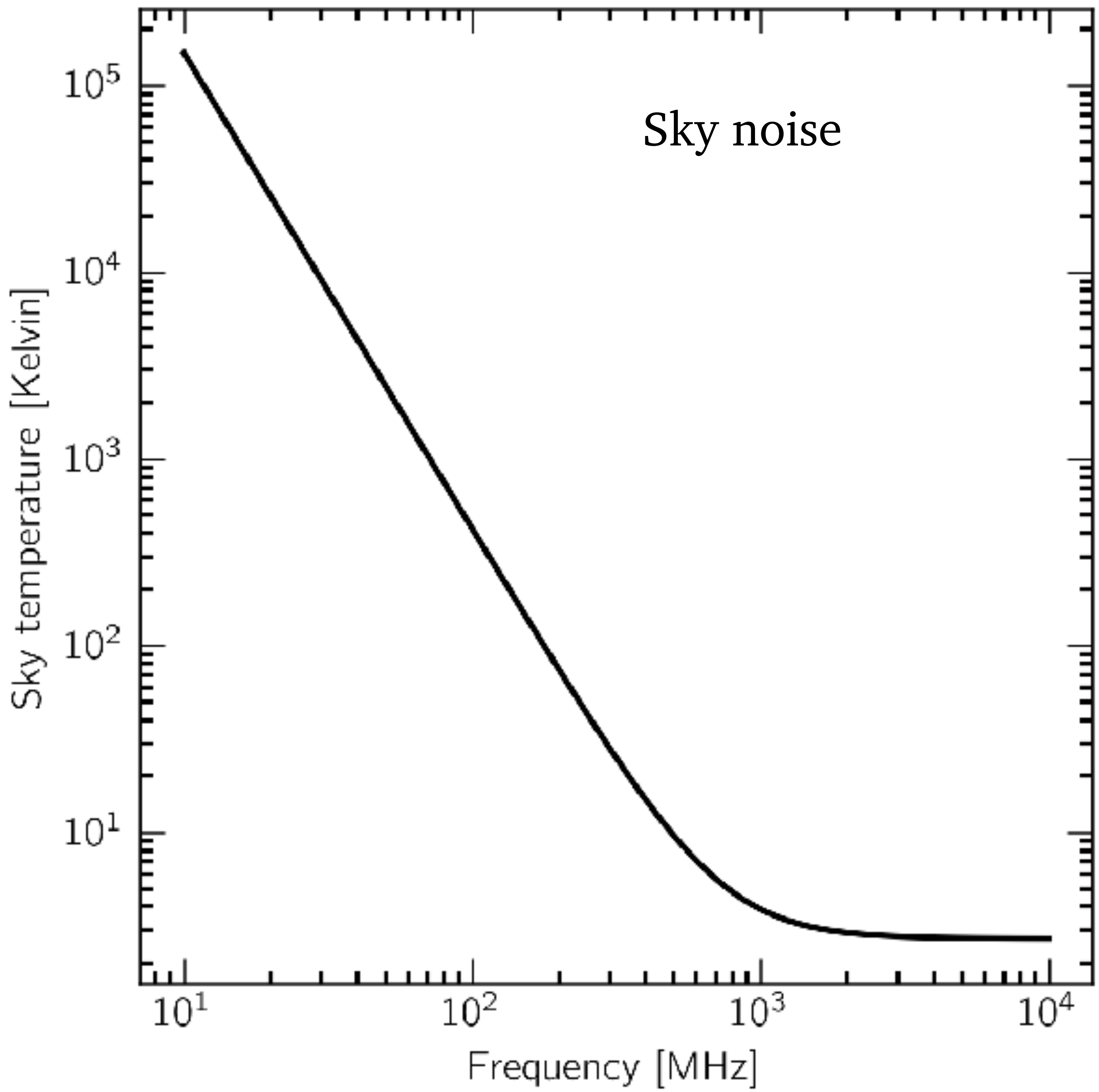
Flux-density is in Jansky units (Jy) : 1 Jaksy $= 10^{-26} \dfrac{\text{Watt}}{m^2 Hz}$

For a given solid-angle flux density can be expressed as a temperature: $\dfrac{2kT}{\lambda^2} = \dfrac{S}{\Omega}$

For the Sun, $\Omega = 0.25\,\text{deg}^2,\ T = 6000\,\text{K},\ \lambda = 2\,\text{m};\ \ S \approx 300\,\text{Jy}$

Appendix: Noise

Thermal noise —> from sky + receiver



Appendix: Noise

System equivalent Flux Density (SEFD): Flux of a source that generates the same power as the noise

$$\frac{2kT_{\rm sys}}{\lambda^2} = \frac{\rm SEFD}{\Omega}$$

$$\mathrm{SEFD} = \frac{2kT_{\rm sys}\Omega}{\lambda^2} = \frac{2kT_{\rm sys}}{A}$$

$$\text{Point-source sensitivity} = \Delta S = \frac{\rm SEFD}{\text{Num . Independent Measurements}}$$

$$\Delta S = \frac{\rm SEFD}{\sqrt{2\Delta\nu\Delta t N_{\rm base}}}$$

(LOFAR station SEFD = 3000 Jy at 150 MHz)

Think of interferometry next time you encounter interference of waves ….